

Mixed Motives and Geometric Representation Theory in Equal Characteristic

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Abstract Let \mathbb{k} be a field of characteristic p . We introduce a formalism of mixed sheaves with coefficients in \mathbb{k} and showcase its use in representation theory.

More precisely, we construct for all quasi-projective schemes $X \rightarrow \overline{\mathbb{F}}_p$ a \mathbb{k} -linear triangulated category of motives on X . Using [Ayo07], [CD12] and [GL00], we show that this system of categories has a six functors formalism and computes higher Chow groups. Indeed, it behaves similarly to other categories of sheaves that one is used to. We attempt to make its construction also accessible to non-experts.

We then consider the subcategory of *stratified mixed Tate motives* defined for affinely stratified varieties X , discuss perverse and parity motives and prove formality results. As an example, we combine these results and [Soe00] to construct a geometric and graded version of Soergel's *modular category* $\mathcal{O}(G)$, consisting of rational representations of a split semisimple group G/\mathbb{k} , and thereby equip it with a full six functor formalism (see [RSW14],[AR16b] for other approaches).

The main idea of using motives in geometric representation theory in this way as well as many results about stratified mixed Tate motives are directly borrowed from Soergel and Wendt [SW16], who tell the story in characteristic zero.

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1 Introduction

Let \mathbb{k} be a field of characteristic p and G/\mathbb{k} be a semisimple simply connected split algebraic group, for example $\mathrm{SL}_n/\mathbb{F}_q$. A fundamental problem in representation theory is to determine the characters of all simple rational G -modules. Unlike in the characteristic zero case, this is still wide open and the subject of ongoing research. In [Soe00], Soergel proposes a strategy using geometric methods: He translates the problem—at least for some of the simple modules—into a question about the geometry of a flag variety X^\vee . He does this by relating certain complexes of sheaves on X^\vee , now known as *parity sheaves* (see [JMW14]), to the projective objects in the *modular category* $\mathcal{O}(G)$ (see Definition 5.1), a subquotient of the category of G -modules. But the beauty and clarity of these and other results in characteristic p geometric representation theory suffer from the lack of an appropriate formalism of *mixed sheaves* with coefficients in \mathbb{k} .

In this article, we construct such a formalism. As an application, we show how to get a purely geometric description of the bounded derived graded category $\mathcal{O}(G)$ in terms of *stratified mixed Tate motives* on X^\vee —the analogue in our formalism of the derived category of constructible sheaves.

Theorem (Theorem 5.4). *Let G be a semisimple simply connected split algebraic group over \mathbb{k} and $X^\vee/\overline{\mathbb{F}}_p$ be the flag variety of the Langlands dual group. Then there is an equivalence of categories*

$$\mathrm{MTDer}_{(B^\vee)}(X^\vee, \mathbb{k}) \xrightarrow{\sim} \mathrm{Der}^b(\mathcal{O}^{\mathbb{Z}, \mathrm{ev}}(G))$$

between the category of stratified mixed Tate motives on X^\vee and the derived evenly graded modular category $\mathcal{O}^{\mathbb{Z}, \mathrm{ev}}(G)$. We have to assume that both the torsion index of G is invertible in \mathbb{F}_p and p is bigger than the Coxeter number of G .

This equips the modular category $\mathcal{O}(G)$ with all the amenities of the geometric world: not only a full six functor formalism $f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om$ (discussed below) à la étale sheaves or mixed Hodge modules, but also, for example, an action of the motivic cohomology operations (in particular the Steenrod algebra), which is of independent interest.

Just as constructible sheaves live inside more general triangulated categories, as for example étale sheaves, our formalism of mixed sheaves is developed using *mixed motives in equal characteristic*, i.e., motives on characteristic p schemes with characteristic p coefficients.

Theorem. *There is a system $\mathrm{H}(X, \mathbb{k})$ of \mathbb{k} -linear tensor triangulated categories of motives associated to quasi-projective schemes $X \rightarrow \overline{\mathbb{F}}_p$. It is equipped with a six functor formalism fulfilling all the usual properties (see Theorem 2.1 for a list). Moreover for smooth varieties $X \rightarrow \overline{\mathbb{F}}_p$ one has*

$$\mathrm{CH}^n(X, 2n-i; \mathbb{k}) \cong \mathrm{Hom}_{\mathrm{H}(X, \mathbb{k})}(\mathbb{1}_X, \mathbb{1}_X(n)[i])$$

where the left hand side denotes higher Chow groups and $\mathbb{1}_X$ is the tensor unit or “constant motive” (see Corollary 2.52).

The category $H(X, \mathbb{k})$ is essentially the *homotopy category of modules over the T -spectrum representing motivic cohomology with \mathbb{k} -coefficients in the Morel–Voevodsky stable homotopy category*, sometimes written as $H\mathbb{k}_X\text{-mod}$ in the motivic literature. However, we have attempted to choose a model as accessible as possible to non-experts. Using Milnor K -theory we avoid any discussion of presheaves with transfers, and using techniques from [CD12] we avoid any mention of simplicial sets or S^1 -spectra, and indeed, even manage to avoid the word T -spectrum. We build our categories step-by-step, from the ground up, using honest sheaves, modules over an explicit monoid object, derived categories of abelian categories, and Verdier quotients, inviting the non-expert to take a peek inside the black box. As the construction is outlined in detail in Section 2.1, we will not say any more about it here.

On our way to the modular category \mathcal{O} , we also discuss weights and categories of weight zero, parity and perverse motives inside $\text{MTDer}_{\mathcal{S}}(X, \mathbb{k})$ for arbitrary affinely stratified varieties X and prove:

Theorem (Corollary 4.5, Theorem 4.11). *Let (X, \mathcal{S}) be an affinely stratified variety over $\overline{\mathbb{F}}_p$ fulfilling some additional conditions—all of them are fulfilled for flag varieties with their Bruhat-stratification. Then there are equivalences of categories*

$$\text{Der}^b(\text{Per}_{\mathcal{S}}(X, \mathbb{k})) \xleftarrow{\sim} \text{MTDer}_{\mathcal{S}}(X, \mathbb{k}) \xrightarrow{\sim} \text{Hot}^b(\text{Par}_{\mathcal{S}}(X, \mathbb{k})_{w=0}) \quad (1)$$

between the derived category of perverse motives, the category of stratified mixed Tate motives, and the homotopy category of weight zero parity motives on X .

The proof heavily relies on the fact that there are no non-trivial extensions between the Tate objects $\mathbb{1}_{\text{Spec}(\overline{\mathbb{F}}_p)}(n)$. That this is true in $H(X, \mathbb{k})$ boils down to a classical observation of Steinberg, namely that the Milnor K -groups $K_n^M(\mathbb{F}_q)$ have no p -torsion for $n > 0$ and q a power of p . This is the reason why, of all things, we work with mixed motives in equal characteristic.

Outline. In Section 2 we construct the system of categories $H(X)$ of motives with coefficients in a commutative \mathbb{Z}/p -algebra A equipped with a six functor formalism. As mentioned before, we go out of our way to make this as accessible as possible to the non-expert.

In Section 3 we define the category of stratified mixed Tate motives as a full subcategory

$$\text{MTDer}_{\mathcal{S}}(X, \mathbb{k}) \subseteq H(X, \mathbb{k}).$$

We consider a weight structure on this category, and prove that $\text{MTDer}_{\mathcal{S}}(X, \mathbb{k})$ is equivalent to the bounded homotopy category of its weight zero objects

$$\text{MTDer}_{\mathcal{S}}(X) \cong \text{Hot}(\text{MTDer}_{\mathcal{S}}(X)_{w=0}), \quad (2)$$

Theorem 3.16. This shows that $\text{MTDer}_{\mathcal{S}}(X)$ is the dg-derived category of a formal (equipped with a trivial differential) graded dg-algebra, Theorem 3.17. In Sec-

tion 3.5 we state the Erweiterungssatz, Theorem 3.20, which implies that the weight zero motives in $\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$ can be realised as graded modules (Soergel modules) over the Chow ring of X , Corollary 3.21

$$\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0} \subseteq \mathrm{CH}^{\bullet}(X, \mathbb{k}(\bullet))\text{-mod}^{\mathbb{Z}}.$$

In Section 4 we study certain interesting subcategories of our category of stratified mixed Tate motives—*parity motives* in Section 4.1 and *perverse motives* in Section 4.2. We also discuss tilting objects and geometric Ringel duality. The general principle here is that everything works as one is used to from constructible étale sheaves or mixed Hodge modules. In Section 4.3 we consider the case of flag varieties.

In Section 5 we apply our results to the representation theory of semisimple algebraic groups in equal characteristic. This is where we observe that we can obtain $\mathcal{O}(G)$ from our categories, Theorem 5.4.

In Section 6 we recall some notions from category theory needed in the construction of $H(X, \mathbb{k})$ for the convenience of the reader.

Future work.

1. In a future paper we extend our results to the equivariant setting and to ind-schemes. Here Iwahori constructible stratified mixed Tate motives on the affine Grassmannian provide a graded version of the derived regular block of the category of all rational representations of G , see [AR16c]. Furthermore, the category of Iwahori equivariant stratified mixed Tate on the affine flag variety is equivalent to the derived Hecke category, corresponding to an affine Weyl group, see [RW16].
2. It would be interesting to study the action of motivic cohomology operations, in particular the Steenrod algebra, on stratified mixed Tate motives with mod- p coefficients. In the case of the flag variety one should obtain a straightforward description of the action on Soergel modules, since they are just constructed from copies of the equivariant motivic cohomology ring $H^{\bullet}_{\mathcal{M}, \mathbb{G}_m}(\mathrm{Spec}(\overline{\mathbb{F}}_p), \mathbb{k}(\bullet)) = \mathbb{k}[u]$, where the Steenrod reduced powers just act by $\mathcal{P}^i(u^k) = \binom{k}{i} u^{k+i(p-1)}$.
3. It would also be interesting to see everything carried out with *integral* coefficients and show that it specialises to Soergel and Wendt's and our construction, for example using the formalism of Spitzweck [Spi12].

Relation to other work.

1. First and foremost, it must be said that this project is strongly inspired by and borrows from Soergel and Wendt's work in characteristic zero, [SW16].
2. The statements in Equations 1 and 2 can also be interpreted as a *formality* result. Namely, that $\mathrm{MTDer}_{\mathcal{S}}(X)$ can be realized as category of dg-modules over a formal dg-algebra. Similar formality results for the flag variety were achieved first by [RSW14] using étale sheaves on X^{\vee}/\mathbb{F}_p with mod- ℓ coefficients, but with a stronger requirement on ℓ , and then by Achar and Riche using their *mixed derived category* (see the next point).

3. In [AR16a] and [AR16b], Achar and Riche present another approach to mixed sheaves. They have the ingenious idea to simply *define* their mixed derived category of a stratified variety X/\mathbb{C} to be the homotopy category of parity sheaves on its complex points $X(\mathbb{C})$, equipped with the metric topology. Equations 1 and 2 imply that this coincides with our category of stratified mixed Tate motives, at least when X is the flag variety—here weight zero parity motives on $X/\overline{\mathbb{F}}_p$ coincide with parity sheaves on $X(\mathbb{C})$. They also reverse engineer some parts of a six functor formalism, as for example base change for locally closed embeddings. On the other hand, our approach has the advantage of being embedded in the rich environment of motives, which immediately implies all those properties and also provides new structures, as for example an action of the motivic cohomology operations. It even yields a sensible outcome for a variety which does not have enough parity motives.
4. The systems of categories $H\mathbb{k}_X\text{-mod}$ are of great interest, and nailing down the six functor formalism à la [Ayo07] for them (for a general ring \mathbb{k}) is one of the achievements of [CD12]. Our contribution is to observe that in our case, Milnor K -theory gives a particularly nice model of $H\mathbb{k}_X$ allowing one to largely avoid the abstract homotopy theory.

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2 Categories of motives and the six operations

In this section we construct a system of categories of motives with coefficients in a commutative \mathbb{Z}/p -algebra A equipped with a six functor formalism. More explicitly, in Definition 2.32 we associate to every quasi-projective variety X/k , for k a perfect field of characteristic p , a symmetric monoidal triangulated category

$$H(X) = H(X, A) = H(X, \mathbb{K} \otimes A) \quad (3)$$

and for every morphism $f : X \rightarrow Y$ a symmetric triangulated functor $f^* : H(Y) \rightarrow H(X)$.

Using a theorem of Geisser–Levine we show in Corollary 2.52 that when X is smooth, the category $H(X)$ can be used to calculate motivic cohomology in Voevodsky’s sense, or equivalently, higher Chow groups, [MVW06, Theorem 19.1], in the sense that there are canonical isomorphisms,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{H}(X,A)}(\mathbb{1}, \mathbb{1}(i)[j]) &\cong H_{\mathcal{M}}^j(X, A(i)) \\ &\cong \mathrm{CH}^i(X, 2i-j) \otimes A. \end{aligned}$$

We observe in Corollary 2.46 that this system of categories is what Ayoub calls a unital symmetric monoidal stable homotopy 2-functor, [Ayo07, Definitions 1.4.1 and 2.3.1], and consequently, satisfies the following list of properties.

Theorem 2.1. *1. For every morphism $f : Y \rightarrow X$ in $\mathrm{QProj}(k)$ the functor f^* has a right adjoint, [Ayo07, Def.1.4.1].*

$$f^* : \mathbf{H}(X) \rightleftarrows \mathbf{H}(Y) : f_*. \quad (4)$$

*2. For any morphism $f : Y \rightarrow X$ in $\mathrm{QProj}(k)$, one can construct a further pair of adjoint functors, the **exceptional functors***

$$f_! : \mathbf{H}(Y) \rightleftarrows \mathbf{H}(X) : f^!$$

which fit together to form a covariant (resp. contravariant) 2-functor $f \mapsto f_!$ (resp. $f \mapsto f^!$), [Ayo07, Prop.1.6.46].

3. For each $X \in \mathrm{QProj}(k)$, the tensor structure on $\mathbf{H}(X)$ is closed in the sense that for every $E \in \mathbf{H}(X)$, the functor $- \otimes E$ has a right adjoint

$$- \otimes E : \mathbf{H}(X) \rightleftarrows \mathbf{H}(X) : \mathcal{H}om_X(E, -), \quad (5)$$

the internal Hom functor.

4. (Stability)¹ For every $X \in \mathrm{QProj}(k)$, let $p : \mathbb{A}_X^1 \rightarrow X$ be the canonical projection with zero section s . Then the endofunctor

$$s^! p^* : \mathbf{H}(X) \rightarrow \mathbf{H}(X) \quad (6)$$

is invertible, [Ayo07, Def.1.4.1]. For $E \in \mathbf{H}(X)$ and $n \in \mathbb{Z}$ we denote

$$E(n) := (s^! p^*)^n(E)[-2n] \quad (7)$$

the n -th Tate twist of E .

5. With X and p as above, \mathbf{H} satisfies \mathbb{A}^1 -homotopy invariance in the sense that the unit of the adjunction (p^, p_*) is an isomorphism, [Ayo07, Def.1.4.1].*

$$\mathrm{id} \xrightarrow{\sim} p_* p^*. \quad (8)$$

6. For any $f : Y \rightarrow X$ in $\mathrm{QProj}(k)$ there exists a natural transformation

$$f_! \rightarrow f_*$$

which is an isomorphism when f is proper, [Ayo07, Def.1.7.1, Thm.1.7.17].

¹ Lemma 2.42 shows that the left adjoint of $s^! p^*$ is isomorphic to $M(\mathbb{G}_m/S) \otimes -$. The term “Stability” is reference to the fact that $\mathbf{H}(S)$ is stable under tensor with $M(\mathbb{G}_m/S)$.

7. (**Relative purity**) For any smooth morphism $f : Y \rightarrow X$ in $\mathbf{QProj}(k)$ of relative dimension d there is a canonical isomorphism, [Ayo07, §1.5.3],

$$f^* \rightarrow f^!(-d)[-2d]. \quad (9)$$

8. (**Base change**) For any cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

there exist natural isomorphisms of functors, [Ayo07, Prop.1.6.48, Chap.1],

$$g^* f_! \xrightarrow{\sim} f'_! g'^*, \quad g'_! f'^! \xrightarrow{\sim} f^! g_*,$$

9. (**Localization**), For $i : Z \rightarrow X$ a closed immersion with open complement $j : U \rightarrow X$, there are distinguished triangles

$$j_! j^! \rightarrow 1 \rightarrow i_* i^* \rightarrow j_! j^![1]$$

$$i_! i^! \rightarrow 1 \rightarrow j_* j^* \rightarrow i_! i^![1]$$

where the first and second maps are the counits and units of the respective adjunctions, [Ayo07, Lem.1.4.6, 1.4.9].

10. (**Projection formulae, Verdier duality**) For any morphism $f : Y \rightarrow X$ in $\mathbf{QProj}(k)$, there exist natural isomorphisms

$$\begin{aligned} (f_! E) \otimes_X F &\xrightarrow{\sim} f_!(E \otimes_Y f^* F), \\ \mathcal{H}om_X(E, f_* F) &\xrightarrow{\sim} f_* \mathcal{H}om_Y(f^* E, F), \\ \mathcal{H}om_X(f_! E, F) &\xrightarrow{\sim} f_* \mathcal{H}om_Y(E, f^! F), \\ f^! \mathcal{H}om_X(E, F) &\xrightarrow{\sim} \mathcal{H}om_Y(f^* E, f^! F). \end{aligned}$$

11. Define the subcategory of **constructible** objects $H^c(S) \subset H(S)$ to be the subcategory of compact objects. This subcategory coincides with the thick full subcategory generated by $f_! f^! \mathbb{1}(n)$ for $n \in \mathbb{Z}$ and $f : X \rightarrow S$ smooth. The six functors $f_!, f^!, f^*, f_*, \otimes, \mathcal{H}om$ preserve compact objects.

12. Let $f : X \rightarrow \mathrm{Spec}(k)$ in $\mathbf{QProj}(k)$. For $E \in H(X)$ we denote by

$$D_X(E) := \mathcal{H}om_X(E, f^! \mathbb{1}) \quad (10)$$

the **Verdier dual** of E . For all $E, F \in H^c(X)$, there is a canonical duality isomorphism

$$D_X(E \otimes D_X(F)) \xrightarrow{\sim} \mathcal{H}om_X(E, F).$$

Furthermore, for any morphism $f : Y \rightarrow X$ in $\mathbf{QProj}(k)$ and any $E \in H^c(X)$ there are natural isomorphisms

$$\begin{aligned}
D_X(D_X(E)) &\cong E, \\
D_Y(f^*(E)) &\cong f^!(D_X(E)), \\
D_X(f_!(E)) &\cong f_*(D_Y(E)).
\end{aligned}$$

Finally, let us mention that $H(-, A)$ is canonically equipped with a morphism from the constant stable homotopy functor with value $D(A\text{-mod})$ the derived category of A -modules. More explicitly, for every $X \in \text{QProj}(k)$ there is a canonical *tensor triangulated* functor, γ , which admits a right adjoint compatible with the f^*

$$\gamma : D(A\text{-mod}) \rightleftarrows H(X) : \Gamma \quad (11)$$

So in particular, we get formulas like

$$H^n \Gamma(\mathcal{H}om_X(E, F)) = \text{Hom}_{H(X)}(E, F[n]). \quad (12)$$

2.1 Construction overview

The construction of H proceeds via a sequence of systems of categories equipped with functorialities (as the base scheme S varies) which are used to build the six operations and force them to have the desired properties. Explicitly, we will end up with a sequence of *morphisms of monoidal Sm -fibered categories*, (i.e., categories equipped with $f_\#, f^*, \otimes$, see Definition 2.6),

$$\begin{array}{ccccc}
\text{Sm}/S & \xrightarrow{\mathbb{Z}_{\text{Nis}}(-)} & \text{Sh}_{\text{Nis}}(\text{Sm}/S) & \longrightarrow & \text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}} \\
M(-) \downarrow & & \searrow \mathbb{K}(-) & & \downarrow \mathbb{K} \otimes^{\mathfrak{S}} - \\
H(S) \stackrel{\text{def}}{=} D(\mathbb{K}_S\text{-mod})/S_{\text{Htp, Stb}} & \longleftarrow & D(\mathbb{K}_S\text{-mod}) & \longleftarrow & \mathbb{K}_S\text{-mod}
\end{array}$$

of which all but Sm/S are *Sm -premotivic categories*, (i.e., categories also equipped with f_* and $\mathcal{H}om$, Definition 2.3).

The system of categories Sm/S (as S varies over quasi-projective k -varieties) is already equipped with functors $f_\#, f^*, \otimes$, (resp. Eq.(14), (15), (16)). However, the right adjoints $f_*, \mathcal{H}om$ are lacking. To obtain these we pass to categories of sheaves, $\text{Sh}_{\text{Nis}}(\text{Sm}/S)$, using the Yoneda embedding, $\mathbb{Z}_{\text{Nis}}(-)$. In order to encourage the Localisation property we use Nisnevich sheaves, cf. Remark 2.10.

The first step to inverting the Tate twist is working with symmetric sequences, $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$, Notation 2.12, cf. Remark 2.13. A symmetric sequence is just a sequence of sheaves $(\mathcal{E}_0, \mathcal{E}_1, \dots)$ equipped with an action of the n -th symmetric group on \mathcal{E}_n . The sheaf \mathcal{E}_n will eventually correspond to $\mathcal{E}_0(n)[n]$, Notation 2.15 and 2.45. The category $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ is equivalent to the category of Nisnevich sheaves on $(\text{Sm}/S) \times (\bigsqcup_{n \geq 0} \mathfrak{S}_n)$ where \mathfrak{S}_n is the n -th symmetric group thought of as a category with one object and $n!$ morphisms. So even though the description may not look like it, $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ is an honest category of sheaves.

Next, we want motivic cohomology to be representable by the twisted, shifted, unit object, so we pass to \mathbb{K} -modules, $\mathbb{K}_S\text{-mod}$, Notation 2.19, cf. Theorem 2.51. To obtain a cohomological framework we use the derived category of \mathbb{K} -modules, $D(\mathbb{K}_S\text{-mod})$, and finally, we impose Homotopy Invariance and invertibility of the Tate twist, using Verdier quotients along the subcategory $\mathcal{S}_{\text{Htp,Stb}}$, corresponding to these properties, Definition 2.32, Notation 2.27, cf. Lemma 2.31, Lemma 2.44. Once we have the functors $f_{\#}, f^*, f_*, \otimes, \mathcal{H}om$ and the properties Homotopy Invariance, Localisation, Stability, we can run Ayoub's machine [Ayo07, Chap.1] to produce the six operations.

2.2 Presheaves and the five operations $f_{\#}, f^*, f_*, \otimes, \mathcal{H}om$.

In this section we define the categories of sheaves we will use to build \mathcal{H} and recall the language of Sm-premotivic categories used to work with the various functoriality properties which will induce those of \mathcal{H} .

Notation 2.2. *We set the following notation.*

- k is a perfect field of positive characteristic p .
- QProj/k is the category of quasi-projective k -varieties.
- Sm is the class of smooth morphisms in QProj/k .
- Sm/S is the category of smooth morphisms $X \rightarrow S$ in QProj/k for S a quasi-projective variety. Morphisms are commutative triangles $Y \rightarrow X \rightarrow S$ in QProj/k (the morphism $Y \rightarrow X$ does not have to be smooth).
- $\text{PSh}(\text{Sm}/S)$ is the category of presheaves of abelian groups on Sm/S where $S \in \text{QProj}/k$.
- $\mathbb{Z}(-)$, the Yoneda embedding (combined with the free abelian presheaf functor)

$$\mathbb{Z}(-) : \text{Sm}/S \rightarrow \text{PSh}(\text{Sm}/S). \quad (13)$$

$\mathbb{Z}(X/Y)$ is the cokernel of the canonical morphism $\mathbb{Z}(Y) \rightarrow \mathbb{Z}(X)$ where $Y \rightarrow X$ is an immersion in Sm/S for some $S \in \text{QProj}/k$.

Notice that the assignment $S \mapsto \text{Sm}/S$ is equipped with functors

$$f^* = T \times_S - : \text{Sm}/S \rightarrow \text{Sm}/T \quad (14)$$

$$\otimes = \times_S : \text{Sm}/S \times \text{Sm}/S \rightarrow \text{Sm}/S \quad (15)$$

$$f_{\#} = \text{Res} : \text{Sm}/T \rightarrow \text{Sm}/S \quad (f : T \rightarrow S \text{ smooth}). \quad (16)$$

However, there is no internal hom, and f^* does not have a right adjoint. To obtain these, we pass to the categories of presheaves using Yoneda. That is, we consider the assignment $S \mapsto \text{PSh}(\text{Sm}/S)$. It is useful to have a name for the structure we obtain.

Definition 2.3 (cf. [CD12] Definitions 1.1.2, 1.1.9, 1.1.10, 1.1.12, 1.1.21, 1.1.27, 1.1.29 and 1.4.2). An Sm-premotivic category on QProj/k is a 2-functor \mathcal{M} , cf.

Def. 6.1, factoring through the category of symmetric monoidal categories, cf. Def. 6.2, satisfying the following properties.

1. **(Adjointness)**

- a. For every morphism $f : T \rightarrow S$ the functor f^* has a right adjoint

$$f^* : \mathcal{M}(S) \rightleftarrows \mathcal{M}(T) : f_* \quad (17)$$

- b. and when f is in Sm , the functor f^* has a left adjoint

$$f_\# : \mathcal{M}(T) \rightleftarrows \mathcal{M}(S) : f^*. \quad (18)$$

If $f = \text{id}$, these are all the identity functors.

2. **(Base change)** Given a cartesian square as below with f (and g) smooth the canonical, cf. [CD12, 1.1.6] and below, natural transformation is an isomorphism.

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ g \downarrow & & \downarrow f \\ T & \xrightarrow{p} & S \end{array} \quad q_\# g^* \xrightarrow{\sim} f^* p_\#$$

3. **(Completeness)** Each of the symmetric monoidal categories $\mathcal{M}(S)$ is complete in the sense that for every object $F \in \text{PSh}(\text{Sm}/S)$ the functor $- \otimes F$ admits a right adjoint

$$- \otimes_S F : \mathcal{M}(S) \rightleftarrows \mathcal{M}(S) : \mathcal{H}om_S(F, -). \quad (19)$$

4. **(Projection formula)** For any smooth morphism f , the canonical, cf. [CD12, 1.1.24], exchange natural transformation

$$f_\#(- \otimes f^*) \rightarrow (f_\#-) \otimes (-) \quad (20)$$

is an isomorphism.

Definition 2.4. A *triangulated Sm-premotivic category* is a Sm-premotivic category taking values in symmetric monoidal triangulated categories, cf. Definition 6.4.

Remark 2.5. The adjective *motivic* is reserved for a premotivic category satisfying Homotopy Invariance, Localisation, Stability, and an Adjoint Property, from which one can then build all the six operations. We will get to all this later. Premotivic categories can be used to construct motivic categories as we will now see.

It is observed in [CD12, Exam.5.1.1] that the assignment sending a variety S in QProj/k to the category $\text{PSh}(\text{Sm}/S)$ and a morphism $f : T \rightarrow S$ in QProj/k to the functor $f^* : \text{PSh}(\text{Sm}/S) \rightarrow \text{PSh}(\text{Sm}/T)$ is an Sm-premotivic category (set $\mathcal{S} = \text{QProj}/k$, $\mathcal{P} = \text{Sm}$, and $\Lambda = \mathbb{Z}$ in their notation). The verification of all these properties for $\text{PSh}(\text{Sm}/-)$ is a formal routine exercise.

Notice that the Yoneda embedding preserves $f_#, f^*, \otimes$. We will also need a name for such a system of natural transformations, and therefore also for an Sm-premotivic category missing f_* and $\mathcal{H}om$.

Definition 2.6 ([CD12, Def.1.1.2, Def.1.1.21, Def.1.1.27], [CD12, Def.1.2.2]). A *monoïdal Sm-fibered category* is a 2-functor \mathcal{M} taking values in symmetric monoïdal categories as in Definition 2.3, satisfying (1b), (2), (4), but not necessarily (1a) or (3).

A *morphism of monoïdal Sm-fibered categories*, $\phi : \mathcal{M} \rightarrow \mathcal{N}$, is the data of a functor $\phi_S : \mathcal{M}(S) \rightarrow \mathcal{N}(S)$ for every $S \in \mathbf{QProj}/k$ and these functors are required to be compatible with the monoïdal Sm-fibered structure in the sense that for every morphism $f : T \rightarrow S$ in \mathbf{QProj}/k there are natural isomorphisms

$$f^* \phi_S \cong \phi_T f^*, \quad \phi_S(- \otimes -) \cong (\phi -) \otimes_S (\phi -), \quad \text{and} \quad f_# \phi_T \cong \phi_S f_# \quad (21)$$

(the latter when f is smooth). These isomorphisms are required to satisfy the appropriate cocycle condition with respect to composition in \mathbf{QProj}/S .

For example, in the case of the Yoneda embedding, one can check that we have

$$f^* \mathbb{Z}(X) \cong \mathbb{Z}(T \times_S X), \quad \mathbb{Z}(X) \otimes \mathbb{Z}(X') \cong \mathbb{Z}(X \times_S X'), \quad \text{and} \quad f_# \mathbb{Z}(Y) \cong \mathbb{Z}(Y) \quad (22)$$

for $f : T \rightarrow S$ a morphism, $X, X' \in \mathbf{Sm}/S$, $Y \in \mathbf{Sm}/T$ and the latter isomorphism only makes sense when f is smooth.

Of course, every Sm-premotivic category is a monoïdal Sm-fibered category. When we talk about morphisms of Sm-premotivic categories, we mean morphisms of their underlying monoïdal Sm-fibered categories.

2.3 Nisnevich sheaves

In this section we pass from presheaves to Nisnevich sheaves.

Definition 2.7. We equip each \mathbf{Sm}/S with the *Nisnevich topology*. Its covering families are those families of étale morphisms $\{U_i \rightarrow X\}_{i \in I}$ which have “constructible” sections, by which we mean there exists a sequence of closed subvarieties $Z_0 \subset Z_1 \subset \dots \subset Z_n = X$ such that for each $j = 1, \dots, n$ there is a factorisation $Z_j - Z_{j-1} \rightarrow U_{i_j} \rightarrow X$ for some $i_j \in I$.

Remark 2.8. By convention one also says the empty set is a covering family of the empty variety. This forces every Nisnevich sheaf F to satisfy $F(\emptyset) = 0$.

Since all étale morphisms are in \mathbf{Sm} , this topology is Sm-admissible in the sense of [CD12, Def.5.1.3].

Notation 2.9. We write

- $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S)$ for the category of sheaves of abelian groups on Sm/S , and
 $\mathbb{Z}_{\mathrm{Nis}}(-)$ when we want to emphasise that the Yoneda embedding $\mathbb{Z}(-)$ takes values in the category of Nisnevich sheaves.
 $\mathbb{Z}_{\mathrm{Nis}}(X/Y)$ As above, when $Y \rightarrow X$ is a immersion in Sm/S for some $S \in \mathrm{QProj}/k$, we write $\mathbb{Z}_{\mathrm{Nis}}(X/Y)$ for the cokernel in the category of Nisnevich sheaves of the canonical morphism $\mathbb{Z}_{\mathrm{Nis}}(Y) \rightarrow \mathbb{Z}_{\mathrm{Nis}}(X)$.

Remark 2.10. Every Zariski covering is a Nisnevich covering, and every Nisnevich covering is an étale covering. Indeed, Morel–Voevodsky use the Nisnevich topology because it captures good properties of both the étale topology and the Zariski topology, [MV99, Sec.3].

1. Like the Zariski topology, for every field k , every Nisnevich sheaf F , and every $i > 0$ we have $H_{\mathrm{Nis}}^i(k, F) = 0$: To show that cohomology vanishes it suffices to show that Čech cohomology vanishes, [Gro72, V.4.3], [Mil80, III.2.11]. Every Nisnevich cover of a field k is refined by the identity cover $\{\mathrm{id} : \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k)\}$, so this cover calculates the Čech cohomology. But the Čech cohomology of this cover is zero.
2. Like the étale topology, closed immersions of smooth varieties look Nisnevich locally like zero sections of vector spaces: If $Z \rightarrow X$ is a closed immersion of smooth varieties of codimension d , every point $z \in Z$ has an open neighbourhood $U \subset X$ in X such that there exists some isomorphism $\mathbb{Z}_{\mathrm{Nis}}(U/U - U \cap Z) \cong \mathbb{Z}_{\mathrm{Nis}}(\mathbb{A}_{U \cap Z}^d / \mathbb{A}_{U \cap Z}^d - s(U \cap Z))$ where $s : U \cap Z \rightarrow \mathbb{A}_{U \cap Z}^d$ is the zero section, [Voe00a, Prop.5.18], [MV99, Proof of Lemma 3.2.28]. This combined with Homotopy Invariance is what gives the Localisation property.
3. Like the in the Zariski topology there is a Mayer-Vietoris property. One can show that a presheaf F is a Nisnevich sheaf if and only if one has $(F(\emptyset) = 0$ and) $F(X) \cong \ker(F(U) \oplus F(V) \rightarrow F(U \times_X V))$ for every cartesian square

$$\begin{array}{ccc} U \times_X V & \xrightarrow{j} & V \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array} \quad (23)$$

such that i is an open immersion, f is a étale morphism, and $f^{-1}(X - U) \rightarrow X - U$ is an isomorphism. Such a square is called a *distinguished Nisnevich square*.

More importantly, we have the following.

Theorem 2.11 ([MV99], [CD12, cf. proof of 3.3.2]). *Let S be a quasi-projective variety and K an (unbounded) complex of Nisnevich sheaves on Sm/S . Then the following two conditions are equivalent.*

1. $H^n(X, K) \rightarrow \mathbb{H}_{\mathrm{Nis}}^n(X, K)$ is an isomorphism for every $X \in \mathrm{Sm}/S, n \in \mathbb{Z}$.

2. $\text{Cone}(K(X) \rightarrow K(U)) \rightarrow \text{Cone}(K(V) \rightarrow K(U \times_X V))$ is a quasi-isomorphism² for every distinguished Nisnevich square (23) (and $K(\emptyset)$ is acyclic).

It is observed in [CD12, Exam.5.1.4] that the assignment $S \mapsto \text{Sh}_{\text{Nis}}(\text{Sm})$ is also a Sm-premotivic category, so we have all the functors, natural transformations, and isomorphisms mentioned in Definition 2.6. Checking this is again a formal routine exercise. The Nisnevich sheaf version of the Yoneda embedding gives us another morphism of monoidal Sm-fibered categories.

$$\mathbb{Z}_{\text{Nis}}(-) : \text{Sm}/S \rightarrow \text{Sh}_{\text{Nis}}(\text{Sm}/S). \quad (24)$$

2.4 Symmetric sequences, the first step towards Stability

In this section we recall what a symmetric sequence is, cf. [HSS00, §2.1], and highlight various functors, i_n, s^n, t^n , which will become important later. This is the first step towards formally inverting $\mathbb{Z}_{\text{Nis}}[\mathbb{G}_m/1]$, cf. Remark 2.13.

Notation 2.12. Consider the following categories.

- \mathfrak{S} will denote the category whose objects are finite (or empty) sets, and morphisms are bijections of sets.
- $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ is the category of symmetric sequences in $\text{Sh}_{\text{Nis}}(\text{Sm}/S)$. This is the category of functors from \mathfrak{S} to $\text{Sh}_{\text{Nis}}(\text{Sm}/S)$. Equivalently, it is the category of sequences of complexes $(\mathcal{E}_0, \mathcal{E}_1, \dots)$ such that each \mathcal{E}_n is equipped with an action of the symmetric group on n letters \mathfrak{S}_n .
- $\otimes^{\mathfrak{S}}$ will be the product on $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ defined as follows. Given symmetric sequences \mathcal{E}, \mathcal{F} the symmetric sequence sends a finite set N to

$$(\mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F})(N) = \bigoplus_{N=P \sqcup Q} \mathcal{E}(P) \otimes \mathcal{F}(Q) \quad (25)$$

where the sum is all decompositions of N into disjoint two finite sets. Given an isomorphism $\phi : N \xrightarrow{\sim} N'$, the morphism $(\mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F})(N) \rightarrow (\mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F})(N')$ is the obvious one induced by the induced isomorphisms $P \xrightarrow{\sim} \phi(P)$ and $Q \xrightarrow{\sim} \phi(Q)$. In the alternative description in which we restrict to the sets $\{1, \dots, n\}$, the product is

$$(\mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F})_n = \bigoplus_{p=0, \dots, n} \text{Ind}_{\mathfrak{S}_p \times \mathfrak{S}_{n-p}}^{\mathfrak{S}_n} \mathcal{E}_p \otimes \mathcal{F}_{n-p} \quad (26)$$

This product is symmetric in the sense that there are canonical functorial isomorphisms $\mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F} \cong \mathcal{F} \otimes^{\mathfrak{S}} \mathcal{E}$.

² This is the formulation we will apply later, but the reader might be more familiar with the equivalent condition: $K(X) \rightarrow K(U) \oplus K(V) \rightarrow K(U \times_X V)$ fits into a distinguished triangle.

$\mathbb{1}^{\mathfrak{S}}$ is the unit for the tensor product $\otimes^{\mathfrak{S}}$. Explicitly, it is the symmetric sequence which sends all nonempty finite sets to 0, and the empty set to the constant sheaf. Equivalently, it is the sequence $(\mathbb{Z}_{\text{Nis}}, 0, 0, \dots)$.

Remark 2.13. Using symmetric sequences is the first step to forcing $\mathbb{Z}_{\text{Nis}}[\mathbb{G}_m/1]$ to be tensor invertible. The n th space \mathcal{E}_n in a symmetric sequence \mathcal{E} represents the n th $\mathbb{Z}_{\text{Nis}}[\mathbb{G}_m/1]$ -loop space, $\mathcal{H}om(\mathbb{Z}_{\text{Nis}}[\mathbb{G}_m/1]^{\otimes n}, \mathcal{E}_0)$, although this only becomes strictly true after we forcibly invert the morphism ϕ_{Stb} in Equation 50.

Remark 2.14. In the sequence formulation, the commutativity isomorphisms of $\otimes^{\mathfrak{S}}$ are a little subtle. This is good motivation for using the “basis-free” definition of symmetric sequences in terms of finite sets. If the reader wishes to check that the canonical monoid structure on $(\mathbb{1}, F, F^{\otimes 2}, F^{\otimes 3}, \dots)$ is commutative (for F a sheaf), see the commutativity isomorphisms, [HSS00, after Proposition 2.1.4].

It is straightforward to check that the structure of Sm -premotivic category on $\text{Sh}_{\text{Nis}}(\text{Sm}/-)$ induces one on $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with the tensor product $\otimes^{\mathfrak{S}}$ and unit $\mathbb{1}^{\mathfrak{S}}$.

Notation 2.15. Let us give some examples of symmetric sequences, and ways of building symmetric sequences out of other ones.

$i_n F$ For any $F \in \text{Sh}_{\text{Nis}}(\text{Sm}/S)$ and $n \geq 0$ consider the sequence which sends a finite set N to 0 if $|N| \neq n$ and $\oplus_{\mathfrak{S}_n} F$ otherwise. As a sequence this looks like

$$i_n F = (0, \dots, 0, \oplus_{\mathfrak{S}_n} F, 0, \dots). \quad (27)$$

The functor i_n is equivalently defined as the left adjoint to the functor taking a symmetric sequence to its n th space

$$i_n : \text{Sh}_{\text{Nis}}(\text{Sm}/S) \rightleftarrows \text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}} : (-)_n. \quad (28)$$

$s^n \mathcal{E}$ Let \mathcal{E} be a symmetric sequence. Define $s\mathcal{E}$ to be the symmetric sequence which sends a finite set N to $\mathcal{E}(N \sqcup \{*\})$. For any $n \geq 0$ we set $s^n \mathcal{E} = \underbrace{s \dots s}_{n \text{ times}} \mathcal{E}$. In the sequence description

$$s^n \mathcal{E} = (\mathcal{E}_n, \mathcal{E}_{n+1}, \mathcal{E}_{n+2}, \dots) \quad (29)$$

where the actions of the \mathfrak{S}_i come from the canonical inclusions $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \dots$

$t^n \mathcal{E}$ The functor s has a left adjoint t which we will also compose with itself to obtain

$$t^n \mathcal{E} = (\underbrace{0, \dots, 0}_n, \text{Ind}_{\mathfrak{S}_0}^{\mathfrak{S}_n} \mathcal{E}_0, \text{Ind}_{\mathfrak{S}_1}^{\mathfrak{S}_{n+1}} \mathcal{E}_1, \text{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_{n+2}} \mathcal{E}_2, \dots) \quad (30)$$

for $n \geq 0$. One checks easily using the fact that $\text{Ind}_{\mathfrak{S}_i}^{\mathfrak{S}_{n+i}}$ is left adjoint to the forgetful functor from \mathfrak{S}_{n+i} objects to \mathfrak{S}_i objects that (t^n, s^n) is an adjunction for all $n \geq 0$.

$$t^n : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S)^{\mathfrak{S}} \rightleftharpoons \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S)^{\mathfrak{S}} : s^n. \quad (31)$$

Moreover, from Equation 26 and 30 we have

$$t\mathcal{E} = \mathcal{E} \otimes t\mathbb{1}^{\mathfrak{S}}. \quad (32)$$

Remark 2.16. The letter t stands for “twist” as t will eventually become isomorphic to the Tate twist $M(\mathbb{Z}_{\mathrm{Nis}}[\mathbb{G}_m]/1) \otimes -$ acting on $H(S)$ and s stands for “detwist”³, it will become isomorphic in $H(S)$ to the inverse of $M(\mathbb{Z}_{\mathrm{Nis}}[\mathbb{G}_m]/\mathbb{Z}_{\mathrm{Nis}}) \otimes -$.

Remark 2.17. Notice that both s and t are exact functors.

2.5 \mathbb{K} -modules, and calculating motivic cohomology

In this section we introduce the symmetric sequence \mathbb{K} representing motivic cohomology (at least for smooth varieties).

Consider \mathbb{G}_m not as a variety to which we can apply $\mathbb{Z}_{\mathrm{Nis}}(-)$, but as the sheaf of groups of invertible global sections $\mathbb{G}_m = \mathcal{O}^* \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S)$. This sheaf sends a smooth S -scheme X to the units of $\Gamma(X, \mathcal{O}_X)$, or equivalently, $\mathrm{Hom}_{\mathrm{Sm}/S}(X, \mathbb{G}_m)$. Recall that the Milnor K -theory $K_{\bullet}^M(R)$ of a ring R is the quotient of the tensor algebra of the units $(R^*)^{\otimes \bullet}$ by the two sided ideal generated by elements of the form $a \otimes (1 - a)$ for $a \in R^* - \{1\}$, [Mil70, §1].

Notation 2.18. Here we define the Milnor K -theory sheaves.

\underline{K}_n^M The sheaf \underline{K}_2^M is defined as the cokernel of the morphism

$$\mathrm{St}_2 : \mathbb{Z}(\mathbb{G}_m - \{1\}) \rightarrow \mathbb{G}_m^{\otimes 2}, \quad \sum n_i a_i \mapsto \sum n_i (a_i \otimes (1 - a_i)). \quad (33)$$

and more generally, for $n > 1$, the sheaf \underline{K}_n^M is defined as the cokernel of the morphism

$$\mathrm{St}_n \stackrel{\mathrm{def}}{=} \sum_{i=0}^{n-2} \mathrm{id}_{\mathbb{G}_m^{\otimes i}} \otimes \mathrm{St}_2 \otimes \mathrm{id}_{\mathbb{G}_m^{\otimes n-i}} : \bigoplus_{i=0}^{n-2} \mathbb{G}_m^{\otimes i} \otimes \mathbb{Z}(\mathbb{G}_m - \{1\}) \otimes \mathbb{G}_m^{\otimes n-i} \rightarrow \mathbb{G}_m^{\otimes n}. \quad (34)$$

\underline{K}_n^M/p Tensoring with \mathbb{Z}/p we obtain the sheaf $\underline{K}_n^M/p = \underline{K}_n^M \otimes \mathbb{Z}/p$.

The interest of these sheaves is that in characteristic p they calculate higher Chow groups, Theorem 2.51.

Notation 2.19. We now define two monoids that interest us.

\mathbb{T} is the commutative monoid

$$\mathbb{T} = (\mathbb{Z}_{\mathrm{Nis}}, \mathbb{Z}_{\mathrm{Nis}}(\mathbb{G}_m/1), \mathbb{Z}_{\mathrm{Nis}}(\mathbb{G}_m/1)^{\otimes 2}, \mathbb{Z}_{\mathrm{Nis}}(\mathbb{G}_m/1)^{\otimes 3}, \dots).$$

³ Actually, s stands for “shift”.

\mathbb{K} The obvious action of \mathfrak{S}_n on $\mathbb{G}_m^{\otimes n}$ descends to an action on \underline{K}_n^M/p and produces a symmetric sequence

$$\mathbb{K} = (\underline{K}_0^M/p, \underline{K}_1^M/p, \underline{K}_2^M/p, \underline{K}_3^M/p, \dots). \\ =_{\mathbb{Z}_{\text{Nis}}/p} \mathcal{O}^*/(\mathcal{O}^*)^p$$

As with \mathbb{T} , the canonical morphisms $(\mathcal{O}^*)^{\otimes N} \otimes (\mathcal{O}^*)^{\otimes M} \rightarrow (\mathcal{O}^*)^{\otimes N \sqcup M}$ induces a morphism of symmetric sequences

$$\mathbb{K} \otimes^{\mathfrak{S}} \mathbb{K} \rightarrow \mathbb{K} \quad (35)$$

compatible with the symmetry isomorphism of $\otimes^{\mathfrak{S}}$ and we obtain a commutative monoid in $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$.

In fact, the canonical morphism $\mathbb{Z}_{\text{Nis}}(\mathbb{G}_m/1) \rightarrow \mathbb{G}_m = \mathcal{O}^*$ induces a morphism of symmetric sequences which gives \mathbb{K} the structure of a \mathbb{T} -algebra.

$$\mathbb{T} \rightarrow \mathbb{K}. \quad (36)$$

$\mathbb{T}_S, \mathbb{K}_S$ If we want to emphasise which base our sheaves are over we will write \mathbb{T}_S and \mathbb{K}_S .

$\mathbb{1}, \mathbb{1}_S$ In a monoidal category, we denote the tensor unit by $\mathbb{1}$. If we want to emphasize that we consider the unit in $\mathbb{R}_S\text{-mod}$, we write $\mathbb{1}_S$.

$\mathbb{R}\text{-mod}$. Given a monoid \mathbb{R} , such as \mathbb{T} or \mathbb{K} , in a monoidal category, such as $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$, we write $\mathbb{R}\text{-mod}$ for the category of its modules.

$\otimes^{\mathbb{R}}$ Given a symmetric monoid \mathbb{R} , the category $\mathbb{R}\text{-mod}$ inherits a canonical structure of symmetric monoidal category. Indeed, for any two \mathbb{R} -modules \mathcal{E}, \mathcal{F} there are two canonical morphisms $\mathcal{E} \otimes^{\mathfrak{S}} \mathbb{R} \otimes^{\mathfrak{S}} \mathcal{F} \rightrightarrows \mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F}$ induced by the \mathbb{R} -module structures $\mu_{\mathcal{E}} : \mathbb{R} \otimes^{\mathfrak{S}} \mathcal{E} \rightarrow \mathcal{E}, \mu_{\mathcal{F}} : \mathbb{R} \otimes^{\mathfrak{S}} \mathcal{F} \rightarrow \mathcal{F}$ of \mathcal{E}, \mathcal{F} respectively, and the symmetry isomorphism $\sigma : \mathcal{E} \otimes^{\mathfrak{S}} \mathbb{R} \rightarrow \mathbb{R} \otimes^{\mathfrak{S}} \mathcal{E}$ of $\otimes^{\mathfrak{S}}$. We define $\otimes^{\mathbb{R}}$ using the coequaliser (i.e., the cokernel of the difference) of these two morphisms

$$\mathcal{E} \otimes^{\mathbb{R}} \mathcal{F} = \text{coker}(\mathcal{E} \otimes^{\mathfrak{S}} \mathbb{R} \otimes^{\mathfrak{S}} \mathcal{F} \xrightarrow{\text{id}_{\mathcal{E}} \otimes^{\mathfrak{S}} \mu_{\mathcal{F}} - (\mu_{\mathcal{E}} \sigma) \otimes^{\mathfrak{S}} \text{id}_{\mathcal{F}}} \mathcal{E} \otimes^{\mathfrak{S}} \mathcal{F}). \quad (37)$$

Ind, Res For any symmetric monoid \mathbb{R} in $\text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ and a symmetric sequence \mathcal{E} we obtain a new symmetric sequence just by tensoring with \mathbb{R} . This new symmetric sequence has a canonical structure of \mathbb{R} -module, induced by the monoid structure of \mathbb{R} . In fact, as one would expect, this process, which we denote $\text{Ind}^{\mathbb{R}}$ is left adjoint to the forgetful functor $\text{Res}_{\mathbb{R}}$. Similarly, if $\mathbb{R} \rightarrow \mathbb{R}'$ is a morphism of symmetric monoids, applying $\mathbb{R}' \otimes^{\mathbb{R}} -$ gives a functor $\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}$ from $\mathbb{R}\text{-mod}$ to $\mathbb{R}'\text{-mod}$. Just as in the classical case, there is a canonical factorisation of the “free-module/forgetful-functor” adjunctions

$$\begin{array}{ccc}
 & \text{Ind}^{\mathbb{R}'} & \\
 & \curvearrowright & \\
 \text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}} & \xrightleftharpoons[\text{Res}_{\mathbb{R}}]{\text{Ind}^{\mathbb{R}}} \mathbb{R}\text{-mod} & \xrightleftharpoons[\text{Res}_{\mathbb{R}'}]{\text{Ind}^{\mathbb{R}'}} \mathbb{R}'\text{-mod} \\
 & \curvearrowleft & \\
 & \text{Res}_{\mathbb{R}'} &
 \end{array} \quad (38)$$

t Note that for any \mathbb{R} -module \mathcal{E} the symmetric sequence $t(\text{Res}_{\mathbb{R}} \mathcal{E})$ comes equipped with a canonical \mathbb{R} -module structure, see Equation 32. That is, the functor t has a canonical extension to $\mathbb{R}\text{-mod}$ compatible with $\text{Res}_{\mathbb{R}}$. Moreover we have

$$\mathcal{E} \otimes^{\mathbb{R}} (t\mathcal{F}) = t(\mathcal{E} \otimes^{\mathbb{R}} \mathcal{F}) \quad (39)$$

for all $\mathcal{E}, \mathcal{F} \in \mathbb{R}\text{-mod}$.

$\mathbb{R}(X)$ Composing all the left adjoints, we find a functor

$$\mathbb{R}(-) : \text{Sm}/S \xrightarrow{\mathbb{Z}_{\text{Nis}}} \text{Shv}_{\text{Nis}}(\text{Sm}/S) \xrightarrow{i_0} \text{Shv}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}} \xrightarrow{\text{Ind}^{\mathbb{R}}} \mathbb{R}\text{-mod}$$

$\mathbb{R}(X/Y)$ If $Y \rightarrow X$ is an embedding of smooth S -varieties we will write $\mathbb{R}(X/Y) = \text{coker}(\mathbb{R}(Y) \rightarrow \mathbb{R}(X))$. In particular, we will be using

$$\mathbb{R}(\mathbb{G}_m/1) = \text{coker}(\mathbb{R}(S) \xrightarrow{1} \mathbb{R}(\mathbb{G}_m)). \quad (40)$$

Not only are \mathbb{T} and \mathbb{K} a symmetric monoid, but the collection of all the $\mathbb{T}_S, \mathbb{K}_S$ for $S \in \text{QProj}/k$ are *cartesian* in the following sense.

Definition 2.20 ([CD12, 1.1.38, 7.2.10]). A *cartesian section* of a monoidal Sm -fibered category \mathcal{M} is a collection of objects $\{A_X \in \mathcal{M}(X)\}_{X \in \text{QProj}/k}$ equipped with an isomorphism $f^*A_X \xrightarrow{\sim} A_Y$ for every $f : Y \rightarrow X \in \text{QProj}/k$ and these isomorphisms are subject to coherence identities, [SGA03, Exp.VI].

A cartesian section $\mathbb{R} = \{\mathbb{R}_X\}$ of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ such that each \mathbb{R}_X is equipped with a monoid structure, and each $f^*\mathbb{R}_X \rightarrow \mathbb{R}_Y$ is a morphism of monoids will be called a *cartesian monoid*. We define similarly a *cartesian commutative monoid*.

Lemma 2.21. For any morphism $f : Y \rightarrow X$ in QProj/k the canonical comparison morphisms $f^*\mathbb{K}_X \rightarrow \mathbb{K}_Y$ are isomorphisms, and make the collection of \mathbb{K}_X a cartesian section. The same is true for \mathbb{T} .

Proof. First we observe that if $W \rightarrow X$ is any smooth scheme, and h_W is the sheaf of sets representing it, then we have $i^*h_W \cong h_{Y \times_X W}$. So in particular, $f^*\mathbb{G}_{m,X} \cong \mathbb{G}_{m,Y}$, and the collection of sheaves \mathbb{G}_m is a cartesian section of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)$. Moreover, f^* preserves \otimes so the collection of $\mathbb{G}_m^{\otimes n}$ is also a cartesian section for any $n \geq 0$. Now recall that \underline{K}_2^M is defined as the cokernel of the morphism St_n , Equation (34). Observing that $f^*\text{St}_{n,X} = \text{St}_{n,Y}$ for any $f : Y \rightarrow X \in \text{QProj}/k$, and f^* preserves cokernels, we find that the collection of \underline{K}_n^M is a cartesian section of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)$. Finally, $\underline{K}_n^M/p = \text{coker}(\underline{K}_n^M \xrightarrow{p} \underline{K}_n^M)$ so, again since f^* preserves cokernels, the collection of \underline{K}_n^M/p is a cartesian section. It follows that \mathbb{K} is a cartesian monoid.

The statement for \mathbb{T} follows directly from \mathbb{Z}_{Nis} being a morphism of monoidal Sm-fibered categories. \square

Cartesianity is of interest as any the assignment $S \mapsto \mathbb{R}_S\text{-mod}$ inherits a canonical structure of Sm-premotivic category whenever \mathbb{R} is a cartesian commutative monoid, [CD12, Prop.5.3.1]. For a morphism $f : T \rightarrow S$ in QProj/k , and $\mathcal{E} \in \mathbb{R}_S\text{-mod}$, the \mathbb{R}_T -module structure on $f^*\mathcal{E}$ come from the \mathbb{R}_S -module structure on \mathcal{E} and the isomorphisms $\mathbb{R}_T \cong f^*\mathbb{R}_S$ and $f^*(-) \otimes f^*(-) \cong f^*(- \otimes -)$. When f is smooth, the \mathbb{R}_S -module structure on $f_*\mathcal{E}$ is induced by the Projection Formula, Definition 2.3(4).

Remark 2.22. It is a straight-forward exercise to check that the functors $\text{Ind}^{\mathbb{R}}$ and $\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}$ of Equation (38) are morphisms of monoidal Sm-fibered categories, and consequently, the functors $\mathbb{R}(-) : \text{Sm}/- \rightarrow \mathbb{R}_-\text{-mod}$ produce a morphism of monoidal Sm-fibered categories.

Remark 2.23. It is also straight-forward to check that the functors Res are exact in the sense that they preserve all limits and colimits (just as in the case of classical rings).

2.6 Derived categories

In this section, we discuss the derived category $D(\mathbb{K}\text{-mod})$ of \mathbb{K} -modules. Our category \mathcal{H} will be the full subcategory of $D(\mathbb{K}\text{-mod})$ of those objects satisfying a stability and homotopy invariance property. We also discuss in this section the interplay between these two properties and the functorialities we have obtained so far.

Notation 2.24. Let $\mathbb{R}_S \in \text{Sh}_{\text{Nis}}(\text{Sm}/S)^{\mathfrak{S}}$ be a symmetric monoid such as \mathbb{K}_S or \mathbb{T}_S and consider the following categories.

- $C(\text{Sh}_{\text{Nis}}(\text{Sm}/S))$ is the category of (unbounded) chain complexes in $\text{Sh}_{\text{Nis}}(\text{Sm}/S)$.
- $C(\mathbb{R}_S\text{-mod})$ is the category of (unbounded) chain complexes in the abelian category $\mathbb{R}_S\text{-mod}$. Note that this is equivalent to the category of \mathbb{R}_S -modules in $C(\text{Sh}_{\text{Nis}}(\text{Sm}/S))^{\mathfrak{S}}$, the category of symmetric sequences in the category of unbounded complexes of Nisnevich sheaves of abelian groups.
- $D(\mathbb{R}_S\text{-mod})$ is the (unbounded) derived category of $\mathbb{R}_S\text{-mod}$. This is not the same as considering \mathbb{R}_S -modules in $D(\text{Sh}_{\text{Nis}}(\text{Sm}/S))^{\mathfrak{S}}$ which in turn is not the same as $D(\text{Sh}_{\text{Nis}}(\text{Sm}/S))^{\mathfrak{S}}$, although there are canonical functors from the formers to the latters.

When \mathbb{R} is cartesian, the systems of categories $C(\mathbb{R}_S\text{-mod})$ and $D(\mathbb{R}_S\text{-mod})$ (as S varies in QProj/k) inherit structures of Sm-premotivic categories. The case $C(\mathbb{R}_S\text{-mod})$ is straight-forward [CD12, Lemma 5.1.7]. The case $D(\mathbb{R}_S\text{-mod})$ uses the theory of descent structures developed [CD09] to observe that the functors $f_{\#}, f^*, f_*, \otimes, \mathcal{H}om$ of the Sm-premotivic category $C(\mathbb{R}_-\text{-mod})$ can be derived,

[CD12, 5.1.16]. The model structure is recalled in Section 6.3 but we will only need the following consequences.

Proposition 2.25. *Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The functor $\mathbb{R}(-)$ composed with the canonical functor $C(\mathbb{R}\text{-mod}) \rightarrow D(\mathbb{R}\text{-mod})$ induces a morphism of monoidal Sm -fibered categories.*

$$\mathbb{R}(-) : \mathrm{Sm}/- \rightarrow D(\mathbb{R}\text{-mod}). \quad (41)$$

That is, for any morphism $T \xrightarrow{f} S \in \mathrm{QProj}/k$, $X, X' \in \mathrm{Sm}/S$, $Y \in \mathrm{Sm}/T$ we have

$$f_{\#}\mathbb{R}(Y) \cong \mathbb{R}(Y) \quad \text{in } D(\mathbb{R}_S\text{-mod}) \text{ (when } f \text{ is smooth),} \quad (42)$$

$$f^*\mathbb{R}(X) \cong \mathbb{R}(T \times_S X) \quad \text{in } D(\mathbb{R}_T\text{-mod}), \text{ and} \quad (43)$$

$$\mathbb{R}(X) \otimes \mathbb{R}(X') \cong \mathbb{R}(X \times_S X') \quad \text{in } D(\mathbb{R}_S\text{-mod}). \quad (44)$$

where the functors $f_{\#}, f^*, \otimes$ on the left are the derived ones acting on $D(\mathbb{R}\text{-mod})$. Moreover, for any $n \geq 0, X \in \mathrm{Sm}/S$ and $\mathcal{E} \in D(\mathbb{R}\text{-mod})$,

$$\mathrm{Hom}_{D(\mathbb{R}\text{-mod})}(t^n \mathbb{R}(X), \mathcal{E}[i]) \cong \mathbb{H}_{\mathrm{Nis}}^i(X, \mathcal{E}_n). \quad (45)$$

where \mathcal{E}_n is the complex obtained by applying the “ n th sheaf” functor $(-)_n$ to the complex of symmetric sequences \mathcal{E} (we forget the \mathbb{R} -module structure).

Proof. The first part comes from the observation that the images of representable sheaves are cofibrant. The second part comes from the definition of fibrancy. See Section 6.3 for these definitions. \square

One of the reasons to use symmetric sequences to invert the Tate twist instead of just a bookkeeping index, is so that our categories admit small sums. This gives us access to the theory of Bousfield localisations/Brown representability à la Neeman. Cf. Section 6.2.

Proposition 2.26. *Let $\mathbb{R} \rightarrow \mathbb{R}'$ be a morphism cartesian commutative monoids of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$, such as $\mathbb{T} \rightarrow \mathbb{K}$. We have the following.*

1. *The forgetful functors of Equation 38 directly pass to the derived categories without having to be derived. On the derived categories they are conservative.*
2. *The category $D(\mathbb{R}\text{-mod})$ admits all small sums and is compactly generated by the objects $t^n \mathbb{R}(X)$ for $X \in \mathrm{Sm}/S$ and $n \geq 0$.*
3. *On the derived categories, all three forgetful functors have left adjoints, and so the adjunctions of Equation 38 induce adjunctions*

$$\begin{array}{ccccc} & & L\mathrm{Ind}^{\mathbb{R}'} & & \\ & \nearrow & & \searrow & \\ D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S)^{\mathfrak{S}}) & \xrightleftharpoons[\mathrm{Res}_{\mathbb{R}}]{L\mathrm{Ind}^{\mathbb{R}}} & D(\mathbb{R}\text{-mod}) & \xrightleftharpoons[\mathrm{Res}_{\mathbb{R}'}]{L\mathrm{Ind}^{\mathbb{R}'}} & D(\mathbb{R}'\text{-mod}). \end{array} \quad (46)$$

$\mathrm{Res}_{\mathbb{R}'}$

4. The functors $L\text{Ind}$ satisfy

$$L\text{Ind}^{\mathbb{R}}(i_n\mathbb{Z}(X)) \cong t^n\mathbb{R}(X), \quad L\text{Ind}^{\mathbb{R}'}(t^n\mathbb{R}(X)) \cong t^n\mathbb{R}'(X) \quad (47)$$

for all $S \in \text{QProj}/S$, $X \in \text{Sm}/S$, $n \geq 0$.

5. The functors $L\text{Ind}$ define morphisms of monoidal Sm -fibered categories.

6. The adjunctions $(L\text{Ind}, \text{Res})$ also satisfy a projection formula: the canonical comparison natural transformation is an isomorphism.

$$L\text{Ind}(- \otimes \text{Res}(-)) \xrightarrow{\sim} L\text{Ind}(-) \otimes (-). \quad (48)$$

Proof. 1. This follows directly from the fact that a morphism being a weak equivalence or not (resp. an object being acyclic or not) has nothing to do with the \mathbb{R} -module structure.

2. Compactness follows from Equation (45): sheaf cohomology commutes with sums (in $\text{Sh}_{\text{Nis}}(\text{Sm}/S)$), the Nisnevich topology has finite cohomological dimension [KS86, §1.2] and so hypercohomology of unbounded complexes also commutes with sums. That it is a generating set (in the sense of Definition 6.8) follows from the above conservativity combined with Equation (45).

3. Here we use Brown representability Theorem 6.9: The forgetful functors commute with products, therefore they admit left adjoints.

4. The second one follows from the first via commutativity of the Diagram 46. By coYoneda , it suffices to show that the two objects in Equation 47 corepresent the same functor. To this end, we observe that there are isomorphisms

$$\begin{aligned} \text{Hom}_{D(\mathbb{R}\text{-mod})}(L\text{Ind}^{\mathbb{R}}(i_n\mathbb{Z}(X)), \mathcal{E}) &\stackrel{\text{adjunction}}{\cong} \text{Hom}_{D(\text{Shv}_{\text{Nis}})^{\infty}}(i_n\mathbb{Z}(X), \text{Res}^{\mathbb{R}}\mathcal{E}) \\ &\cong \mathbb{H}_{\text{Nis}}^0(X, \mathcal{E}_n) \stackrel{\text{Eq. (45)}}{\cong} \text{Hom}_{D(\mathbb{R}\text{-mod})}(t^n\mathbb{R}(X), \mathcal{E}), \end{aligned}$$

all functorial in \mathcal{E} .

5. Let $f : T \rightarrow S$ be a morphism in QProj/k . The functors $f_{\#}$ (when f is smooth), f^* , \otimes , and $L\text{Ind}$ are all left adjoints, and therefore commute with sums, and our categories are compactly generated, so to show some compatibility relation such as $f^*L\text{Ind} \cong L\text{Ind}f^*$, it suffices to check it on a set of compact generators. But we have just seen that on our set of compact generators, the functor $L\text{Ind}$ acts as the underived Ind . Moreover, the functors $f_{\#}$ (when f is smooth), f^* , \otimes , and t also act on our compact generators as their underived versions; Proposition 2.25, Remark 2.17. So $L\text{Ind}$ being a morphism of monoidal Sm -fibered categories follows from Ind being a morphism of monoidal Sm -fibered categories, Remark 2.22.
6. Since all functors in question are triangulated and preserve sums, Remark 2.23, it suffices to check the morphism on compact generators. We want to show that $L\text{Ind}(t^n\mathbb{R}(X) \otimes \text{Res}(t^m\mathbb{R}'(Y))) \rightarrow L\text{Ind}(t^n\mathbb{R}(X)) \otimes t^m\mathbb{R}'(Y)$ is an isomorphism for all $X, Y \in \text{Sm}/S$ and $n, m \geq 0$. This follows from Equation 47 and the fact that $L\text{Ind}$ preserves \otimes . \square

Notation 2.27. Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. Consider now the following subcategories of $D(\mathbb{R}\text{-mod})$, cf. [CD12, 5.2.15, 5.3.21].

$\mathcal{S}_{\mathrm{Htp}}$ is defined to be the “ t -stable” thick tensor ideal of $D(\mathbb{R}\text{-mod})$ generated by the cone of

$$\phi_{\mathrm{Htp}} : \mathbb{R}(\mathbb{A}_S^1) \rightarrow \mathbb{R}(S). \quad (49)$$

That is, it is the smallest full subcategory of $D(\mathbb{R}\text{-mod})$ containing this cone which is triangulated, closed under direct summands and all small direct sums (thick), and satisfies the properties $\mathcal{S}_{\mathrm{Htp}} \otimes \mathcal{E} \subseteq \mathcal{S}_{\mathrm{Htp}}$ for any $\mathcal{E} \in D(\mathbb{R}\text{-mod})$ (ideal), and $t\mathcal{S}_{\mathrm{Htp}} \subseteq \mathcal{S}_{\mathrm{Htp}}$ (t -stable).

Suppose that \mathbb{R} is equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$.

$\mathcal{S}_{\mathrm{Stb}}$ is the t -stable thick tensor ideal of $D(\mathbb{R}\text{-mod})$ generated by the cone of the morphism

$$\phi_{\mathrm{Stb}} : t\mathbb{R}(\mathbb{G}_m/1) \rightarrow \mathbb{R}(S) \quad (50)$$

which we will now describe. Consider the adjunction $\mathbb{T} \otimes^{\mathfrak{S}} i_1 : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S) \rightleftarrows \mathbb{T}_S\text{-mod} : (-)_1$ whose right adjoint sends a \mathbb{T} -module to the first (not zeroth) sheaf in the underlying symmetric sequence. For $\mathbb{R} = \mathbb{T}$ we define ϕ_{Stb} to be the counit of the adjunction $(\mathbb{T} \otimes^{\mathfrak{S}} i_1) \circ (-)_1(\mathbb{T}) \cong t\mathbb{T}(\mathbb{G}_m/1) \rightarrow \mathbb{T}$, and otherwise we apply $\mathrm{Lnd}_{\mathbb{T}}^{\mathbb{R}}$ to the ϕ_{Stb} for \mathbb{T} . It follows from Equation (47) that we get a morphism between the appropriate objects.

$\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$ is the smallest t -stable thick tensor ideal of $D(\mathbb{R}\text{-mod})$ generated by the two morphisms (49) and (50).

$\mathcal{S}_{*, S}$ If we want to emphasise the base variety S we will use this notation for \mathcal{S}_* .

$\mathcal{S}_{*, S}^{\mathbb{R}}$ We will use this notation if we want to emphasise the monoid \mathbb{R} .

Lemma 2.28. Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The category $\mathcal{S}_{\mathrm{Htp}}$ is equivalently defined as the smallest triangulated subcategory of $D(\mathbb{R}\text{-mod})$ closed under small sums and containing the cones of $\phi_{\mathrm{Htp}} \otimes t^n \mathbb{R}(X)$ for all $n \geq 0, X \in \mathrm{Sm}/S$. If \mathbb{R} is equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, the analogous statement is true of $\mathcal{S}_{\mathrm{Stb}}$ and $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$.

Proof. This follows directly from the fact that $D(\mathbb{R}\text{-mod})$ is the smallest triangulated subcategory of $D(\mathbb{R}\text{-mod})$ closed under small sums and containing the $t^n \mathbb{R}(X)$ for $n \geq 0, X \in \mathrm{Sm}/S$, Theorem 6.7, Proposition 2.26(2). \square

Lemma 2.29. Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. For any morphism of varieties $f : T \rightarrow S$ in QProj/k , we have $f^* \mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}, S} \subseteq \mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}, T}$. If f is smooth we have $f_{\#} \mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}, T} \subseteq \mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}, S}$.

Proof. Since f^* preserves \otimes , by the definition of $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$ as an ideal it suffices to show that

$$f^*(\phi_{*, S}) = \phi_{*, T}, \quad \text{for } * = \mathrm{Htp}, \mathrm{Stb}. \quad (51)$$

For Htp this follows from $\mathbb{R}(-)$ commuting with f^* , Proposition 2.25. The case of Stb also follows from this, together with f^* commuting with the adjunction $(\mathbb{T} \otimes^{\mathfrak{S}} i_1, (-)_1)$ and the functor $L\text{Ind}_{\mathbb{T}}^{\mathbb{R}}$, it is just a little fiddly.

For $f_{\#}$, by Lemma 2.28, we should show that $f_{\#}$ sends $\phi_{*,T} \otimes t''\mathbb{R}(X)$ inside $\mathcal{S}_{*,S}$ for $*$ = Htp, Stb . This follows from the Projection Formula, Definition 2.3(4), and Equation 51. \square

Lemma 2.30. *Let \mathbb{R} and \mathbb{R}' be cartesian commutative monoids in $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with morphisms of monoids $\mathbb{T} \rightarrow \mathbb{R} \rightarrow \mathbb{R}'$. For any $S \in \text{QProj}/k$ we have*

$$L\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}(\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}}) \subseteq \mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}'} \quad \text{and} \quad \text{Res}_{\mathbb{R}'}^{\mathbb{R}}(\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}'} \subseteq \mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}}. \quad (52)$$

Proof. For the first claim, since $L\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}$ preserves \otimes , Proposition 2.26(5), and the $\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}}$ are defined as ideals, it suffices to show that $L\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}$ sends $\phi_{\text{Htp}}^{\mathbb{R}}$ and $\phi_{\text{Stb}}^{\mathbb{R}}$ inside $\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}'}$. One checks that, in fact, we have

$$L\text{Ind}_{\mathbb{R}}^{\mathbb{R}'}(\phi_*^{\mathbb{R}}) \cong \phi_*^{\mathbb{R}'}, \quad * = \text{Htp}, \text{Stb}. \quad (53)$$

For the second claim we use the alternative definition of Lemma 2.28. We must show that $\text{Res}_{\mathbb{R}'}^{\mathbb{R}}(\phi_*^{\mathbb{R}'} \otimes t''\mathbb{R}'(X))$ is in $\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}}$ for $*$ = Htp, Stb . This follows from Equation 53 and Equation 48. \square

Lemma 2.31. *Let \mathbb{R} be a cartesian commutative monoid of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. An object $\mathcal{E} \in D(\mathbb{R}\text{-mod})$ is $\mathcal{S}_{\text{Stb}}^{\mathbb{R}}$ -local (i.e., $\text{Hom}_{D(\mathbb{R}\text{-mod})}(-, \mathcal{E})$ sends objects of $\mathcal{S}_{\text{Stb}}^{\mathbb{R}}$ to zero) if and only if for every $X \in \text{Sm}/S$, $n, i \geq 0$ the morphisms*

$$\mathbb{H}_{\text{Nis}}^i(X, \mathcal{E}_n) \rightarrow \ker \left(\mathbb{H}_{\text{Nis}}^i(\mathbb{G}_m \times_S X, \mathcal{E}_{n+1}) \rightarrow \mathbb{H}_{\text{Nis}}^i(X, \mathcal{E}_{n+1}) \right) \quad (54)$$

corresponding to

$$\text{Hom}_{D(\mathbb{R}\text{-mod})}((\phi_{\text{Stb}}) \otimes t''\mathbb{R}(X), \mathcal{E}) \quad (55)$$

under Equation 45 are all isomorphisms, where ϕ_{Stb} is the morphism from Equation 50. Similarly, an object is $\mathcal{S}_{\text{Htp}}^{\mathbb{R}}$ -local if and only if for all $X \in \text{Sm}/S$, $n \geq 0, i \in \mathbb{Z}$, the canonical morphisms

$$\mathbb{H}_{\text{Nis}}^i(X, \mathcal{E}_n) \rightarrow \mathbb{H}_{\text{Nis}}^i(\mathbb{A}_X^1, \mathcal{E}_n) \quad (56)$$

are isomorphisms, and an object is $\mathcal{S}_{\text{Htp}, \text{Stb}}^{\mathbb{R}}$ -local if and only if it is both $\mathcal{S}_{\text{Htp}}^{\mathbb{R}}$ -local and $\mathcal{S}_{\text{Stb}}^{\mathbb{R}}$ -local.

Proof. This follows directly from the description of Lemma 2.28 and Proposition 2.25(44). \square

2.7 Motives—definitions

In this section we complete our construction of H .

Definition 2.32. Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. For $S \in \mathrm{QProj}/k$ define

$$H(S, \mathbb{R}) \stackrel{\mathrm{def}}{=} D(\mathbb{R}_S\text{-mod})/\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}. \quad (57)$$

See Section 6.2 for some rappels about the Verdier quotient appearing in Equation 57.

Remark 2.33. In the case $\mathbb{R} = \mathbb{T}$, Cisinski–Déglise use the following notation, [CD12, 5.3.21, Def.5.3.22, Exam.5.3.31, Rem.5.3.34]

$$D_{\mathrm{A}^1}(S) \stackrel{\mathrm{def}}{=} H(S, \mathbb{T}). \quad (58)$$

Proposition 2.34. Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The system of categories $H(-, \mathbb{R})$ inherit a structure of triangulated Sm -premotivic category from $D(\mathbb{R}_S\text{-mod})$, cf. Definition 2.4. Moreover, the canonical functors

$$D(\mathbb{R}_S\text{-mod}) \rightarrow H(S, \mathbb{R}) \quad (59)$$

form a morphism of Sm -premotivic categories in the sense that they commute (up to canonical isomorphism) with the f^* the \otimes , and for smooth morphisms, the $f_{\#}$. In particular, the composition

$$M(-) : \mathrm{Sm}/S \rightarrow D(\mathbb{R}_S\text{-mod}) \rightarrow H(S) \quad (60)$$

is a morphism of monoidal Sm -fibered categories; for any $f : S' \rightarrow S$ in QProj/k , $X, Y \in \mathrm{Sm}/S$, and $X' \in \mathrm{Sm}/S'$, we have

$$f^*M(X) \cong M(S' \times_S X) \quad (61)$$

$$M(X) \otimes M(Y) \cong M(X \times_S Y) \quad (62)$$

$$f_{\#}M(X') \cong M(X') \quad (f \text{ smooth}) \quad (63)$$

Moreover, $H(S)$ is compactly generated by the $t^n M(X)$ for $n \geq 0, X \in \mathrm{Sm}/S$.

Proof. Let $f : T \rightarrow S$ be a morphism in QProj/k . By Lemma 2.29, and the description of $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$ as an ideal, the functors $f_{\#}$ (if f is smooth), f^* , \otimes preserve $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$. Hence, they descend to the Verdier quotient by the universality in the definition, cf. Def. 6.11. The statement about being compactly generated follows directly from the fact that $D(\mathbb{R}\text{-mod})$ is compactly generated by these objects, and the adjunction of Rappels 6.12(2). \square

Notation 2.35. We have defined the composition $\mathrm{Sm}/S \rightarrow D(\mathbb{R}_S\text{-mod}) \rightarrow H(S, \mathbb{R})$ of the canonical functors as

$M(-) : \mathbf{Sm}/S \rightarrow \mathbf{H}(S, \mathbb{R})$.

Remark 2.36. For any smooth morphism $f : X \rightarrow S$ and $\mathcal{E} \in \mathbf{H}(S, \mathbb{R})$ by the Projection Formula, Definition 2.3(4), we have isomorphisms $f_{\#}f^*(\mathcal{E}) \cong f_{\#}f^*(1 \otimes \mathcal{E}) \cong f_{\#}f^*(1) \otimes \mathcal{E}$ functorial in \mathcal{E} . We also have $f_{\#}f^*(1) \cong f_{\#}f^*M(S) \cong M(X)$ by M being a morphism of monoidal \mathbf{Sm} -fibered categories. Consequently, we have a natural isomorphisms

$$f_{\#}f^*(-) \cong M(X) \otimes (-) \quad (64)$$

functorial in \mathbf{Sm}/S .

Proposition 2.37. *Let \mathbb{R}, \mathbb{R}' be cartesian commutative monoids of $\mathbf{Sh}_{\mathbf{Nis}}(\mathbf{Sm}/-)^{\mathfrak{S}}$ equipped with morphisms of monoids $\mathbb{T} \rightarrow \mathbb{R} \rightarrow \mathbb{R}'$. The adjunction $(L\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}'}, \mathrm{Res}_{\mathbb{R}'}^{\mathbb{R}})$ of Equation 46 passes to the Verdier quotients.*

$$\mathbf{H}(S, \mathbb{R}) \rightleftarrows \mathbf{H}(S, \mathbb{R}') \quad (65)$$

1. *The left adjoint is a morphism of \mathbf{Sm} -premotivic categories.*
2. *The right adjoint commutes with f^* for any morphism $f \in \mathbf{QProj}/k$ (but it is not a morphism of monoidal \mathbf{Sm} -fibered categories).*
3. *For every $S \in \mathbf{QProj}/k$, the functor right adjoint is conservative.*

Proof. All claims except the last one follow from the universal property of Verdier localisations, Def. 6.11, since the subcategories $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$ are preserved by the functors in question. The last claim follows from the description of $\mathbf{H}(S, \mathbb{R})$ and $\mathbf{H}(S, \mathbb{R}')$ as full subcategories of $D(\mathbb{R}\text{-mod})$ and $D(\mathbb{R}'\text{-mod})$; the functor $\mathrm{Res}_{\mathbb{R}'}^{\mathbb{R}}$ preserves these full subcategories because its left adjoint $L\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}'}$ preserves the $\mathcal{S}_{\mathrm{Htp}, \mathrm{Stb}}$. \square

2.8 Motives—properties

In this section we develop the properties of $\mathbf{H}(-)$ that we are interested in.

Proposition 2.38. *Let \mathbb{R} be a cartesian commutative monoid of $\mathbf{Sh}_{\mathbf{Nis}}(\mathbf{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. For any morphism $f : T \rightarrow S \in \mathbf{QProj}/k$, the functor $f_* : \mathbf{H}(T, \mathbb{R}) \rightarrow \mathbf{H}(S, \mathbb{R})$ admits a right adjoint.*

Proof. By Brown representability, Theorem 6.9, it suffices to show that f_* preserves small sums. Since we are working with compactly generated categories, Proposition 2.34, the right adjoint f^* preserves small sums if and only if its left adjoint f^* preserves compact objects, [Nee96, Thm.5.1]. In fact, it suffices that f^* sends each compact generator $t^n M(X), X \in \mathbf{Sm}/S, n \geq 0$ to a compact object. But since M and t^n are morphisms of monoidal \mathbf{Sm} -fibered categories, we have $f^*t^n M(X) \cong t^n M(T \times_S X)$. \square

Proposition 2.39 (Homotopy invariance). *Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The Sm -premotivic category $\mathrm{H}(-, \mathbb{R})$ is \mathbb{A}^1 -homotopy invariant in the sense that for any $S \in \mathrm{QProj}/k$ the unit of adjunction $\mathrm{id} \rightarrow p_* p^*$ is an isomorphism where $p : \mathbb{A}_S^1 \rightarrow S$ is the canonical projection.*

Proof. By adjunction it suffices to prove that $p_{\#} p^* \rightarrow \mathrm{id}$ is an equivalence. Both $p_{\#}$ and p^* are left adjoints, and consequently commute with all small sums. So since the category $\mathrm{H}(S)$ is compactly generated by the $t^n M(X)$ for $n \geq 0$ and $X \in \mathrm{Sm}/S$, Proposition 2.34, it suffices to show that the morphisms $p_{\#} p^* t^n M(X) \rightarrow t^n M(X)$ are all isomorphisms. But these are isomorphic to the images of the morphisms $\phi_{\mathrm{Htp}} \otimes t^n \mathbb{R}(X)$ by Proposition 2.25 and these are isomorphisms by definition since they are used to define $\mathcal{S}_{\mathrm{Htp}}$, Lemma 2.28. \square

Proposition 2.40 (Localisation). *Let \mathbb{R} be a cartesian commutative monoid of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. For any closed immersion $i : Z \rightarrow X$ in QProj/k with open complement $j : U \rightarrow X$ the pair*

$$(i^*, j^*) : \mathrm{H}(X, \mathbb{R}) \rightarrow \mathrm{H}(Z, \mathbb{R}) \times \mathrm{H}(U, \mathbb{R}) \quad (66)$$

is conservative, and the counit $i^ i_* \rightarrow \mathrm{id}$ is an isomorphism.*

Proof. Rather than reproduce the proof of [MV99, Theorem 3.2.21] as most writers do, we will deduce localisation for $\mathrm{H}(-, -R)$ from localisation for $D_{\mathbb{A}^1} = \mathrm{H}(-, \mathbb{T})$. Indeed, $D_{\mathbb{A}^1}$ satisfies Localisation, [CD12, Theorem 6.2.1]. Since the right adjoint from Proposition 2.37 is conservative and commutes with j^* , i^* , and i_* , it follows that $\mathrm{H}(-, \mathbb{R})$ also satisfies Localisation. \square

Corollary 2.41. *With notation as in Proposition 2.40 the unit and counit of adjunction fit into a distinguished triangle*

$$j_{\#} j^* \rightarrow \mathrm{id} \rightarrow i_* i^* \rightarrow j_{\#} j^*[1] \quad (67)$$

Proof. First notice that we have the identities

$$i^* i_* i^* \cong i^*, \quad i^* j_{\#} \cong 0, \quad j^* j_{\#} \cong \mathrm{id}, \quad j^* i_* \cong 0. \quad (68)$$

The first one is part of Localisation. The second two follow from the observation that $\mathrm{H}(\emptyset)$ is the zero category and the Base Change property, Definition 2.3(2). The last one follows from the fact that $j^* i_*$ is the right adjoint to $i^* j_{\#}$. For any object \mathcal{E} , choose a cone \mathcal{F} of $j_{\#} j^* \mathcal{E} \rightarrow \mathcal{E}$. It follows from adjunction and $j^* i_* \cong 0$ that $\mathrm{Hom}(j_{\#} j^* \mathcal{E}, i_* i^* \mathcal{E}) = 0$ and so by the usual triangulated category long exact Hom sequence, there exists a factorisation $j_{\#} j^* \mathcal{E} \rightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \rightarrow i_* i^* \mathcal{E}$. Applying i^* to these morphisms and using the identities (68) we obtain $0 \rightarrow i^* \mathcal{E} \rightarrow i^* \mathcal{F} \rightarrow i^* \mathcal{E}$ which shows that $i^* \phi$ is an isomorphism. Applying j^* to these morphisms and using the identities (68) we obtain $j^* \mathcal{E} \rightarrow j^* \mathcal{E} \rightarrow j^* \mathcal{F} \rightarrow 0$ which shows that $j^* \phi$ is an

isomorphism. Hence, since (j^*, i^*) is conservative, it follows that ϕ is an isomorphism. Moreover, applying $\text{Hom}(j_{\#}j^*\mathcal{E}[1], i_*i^*\mathcal{E}) = 0$, we see that \mathcal{F} was unique up to unique isomorphism. \square

Lemma 2.42. *Let \mathbb{R} be a cartesian commutative monoid of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. There is a natural isomorphism of endofunctors*

$$p_{\#}s_*(-) \cong - \otimes M(\mathbb{G}_m/S)[1] : \text{H}(S, \mathbb{R}) \rightarrow \text{H}(S, \mathbb{R}) \quad (69)$$

where $p : \mathbb{A}_S^1 \rightarrow S$ is the canonical projection and $s : S \rightarrow \mathbb{A}_S^1$ the zero section. Consequently, there is also a natural isomorphism of their right adjoints

$$s^!p^*(-) \cong \mathcal{H}om_{\text{H}(S, \mathbb{R})}(M(\mathbb{G}_m/S)[1], -) \quad (70)$$

where $s^!$ is the right adjoint to s_* given by Proposition 2.38, cf. Theorem 2.1(6).

Proof. Consider the localisation triangles, Equation 67, associated to the open immersion $j : \mathbb{G}_m \rightarrow \mathbb{A}^1$. This gives rise to distinguished triangles

$$p_{\#}(j_{\#}j^*)p^* \rightarrow p_{\#}p^* \rightarrow p_{\#}(s_*s^*)p^* \rightarrow . \quad (71)$$

We have $p_{\#}p^* \cong \text{id}$ by homotopy invariance, Proposition 2.39, and $p_{\#}(j_{\#}j^*)p^*(-) \cong - \otimes M(\mathbb{G}_m)$ by Equation 64, so our distinguished triangle becomes

$$(- \otimes M(\mathbb{G}_m)) \rightarrow (- \otimes M(S)) \rightarrow p_{\#}s_* \rightarrow , \quad (72)$$

The structural morphism $M(\mathbb{G}_m) \rightarrow M(S)$ is split by the identity section, and so $M(\mathbb{G}_m)$ decomposes as a direct sum $M(\mathbb{G}_m) \cong M(\mathbb{G}_m/1) \oplus M(S)$, giving an isomorphism $M(\mathbb{G}_m/1)[1] \cong \text{Cone}(M(\mathbb{G}_m) \rightarrow M(S))$. The result follows. \square

Proposition 2.43 (Stability). *Let \mathbb{R} be a cartesian commutative monoid of the category $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The Sm -premotivic category $\text{H}(-, \mathbb{R})$ is stable in the following sense. For any $S \in \text{QProj}/k$ the endofunctor $s^!p^*$ is an equivalence where $p : \mathbb{A}_S^1 \rightarrow S$ is the canonical projection, $s : S \rightarrow \mathbb{A}_S^1$ the zero section, and $s^!$ is the right adjoint to s_* , cf. Proposition 2.38.*

Proof. By Lemma 2.42 it suffices to show that $\mathcal{H}om_{\text{H}(S, \mathbb{R})}(M(\mathbb{G}_m/S), -)$ is an invertible endofunctor. We do this in Lemma 2.44. \square

Lemma 2.44. *The functor $\mathcal{H}om_{\text{H}(S)}(M(\mathbb{G}_m/1), -)$ on $\text{H}(S, \mathbb{R})$ is an equivalence, inverse to the functor induced by $s : (\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \dots) \mapsto (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots)$. Consequently, there are natural isomorphisms of endofunctors*

$$t \cong \mathcal{H}om_{\text{H}(S)}(M(\mathbb{G}_m/1), -), \quad s \cong - \otimes M(\mathbb{G}_m/1). \quad (73)$$

Proof. To evaluate these two functors on $H(S, \mathbb{R})$, we consider the full subcategory of $D(\mathbb{R}\text{-mod})$ consisting of $\mathcal{S}_{\text{Htp, Stb}}$ -local objects, Rappels 6.12(3), which are fibrant, Definition 6.15. Fibrancy, and \mathcal{S}_{Stb} -locality imply that for any such object, and any $X \in \text{Sm}/S$, the canonical morphism

$$H^i(X, \mathcal{E}_n) \rightarrow H^i(X, \mathcal{H}om_{C(\text{Sh}_{\text{Nis}}(\text{Sm}/S))}(\mathbb{Z}_{\text{Nis}}(\mathbb{G}_m/1), \mathcal{E}_{n+1})) \quad (74)$$

is an isomorphism, Lemma 2.31. That is, the canonical morphism

$$\mathcal{E}_n \rightarrow \mathcal{H}om_{C(\text{Sh}_{\text{Nis}}(\text{Sm}/S))}(\mathbb{Z}_{\text{Nis}}(\mathbb{G}_m/1), \mathcal{E}_{n+1}) \quad (75)$$

is a quasi-isomorphism or in other words the morphism

$$\mathcal{E} \rightarrow \mathcal{H}om_{C(\mathbb{R}\text{-mod})}(\mathbb{R}(\mathbb{G}_m/1), s\mathcal{E}) = s\mathcal{H}om_{C(\mathbb{R}\text{-mod})}(\mathbb{R}(\mathbb{G}_m/1), \mathcal{E}) \quad (76)$$

is a quasi-isomorphism, and therefore the corresponding morphism in $D(\mathbb{R}\text{-mod})$ is an isomorphism. So we have shown that we have natural isomorphisms of endofunctors of $H(S, \mathbb{R})$

$$\text{id} \cong s \circ \mathcal{H}om_{H(S, \mathbb{R})}(M(\mathbb{G}_m/1), -) \cong \mathcal{H}om_{H(S, \mathbb{R})}(M(\mathbb{G}_m/1), -) \circ s \quad (77)$$

from which it follows that both s and $\mathcal{H}om_{H(S, \mathbb{R})}(M(\mathbb{G}_m/1), -)$ are essentially surjective and fully faithful, and inverse equivalences of categories. The “Consequently” statement follows directly by uniqueness of adjoints. \square

Notation 2.45. Recall that for $n \in \mathbb{Z}$ we have defined

$$(-)(n) : H(S, \mathbb{R}) \rightarrow H(S, \mathbb{R}) \text{ to be } (s^! p^*)^n(-)[-2n]$$

where $p : \mathbb{A}_S^1 \rightarrow S$ is the canonical projection, $s : S \rightarrow \mathbb{A}_S^1$ the zero section, and $s^!$ is the right adjoint to s_* , Theorem 2.1(2), (6), (4). By Lemmas 2.42 and 2.44 we see that we have when $n \geq 0$ we have

$$(-)(-n)[-n] \cong \mathcal{H}om(M(\mathbb{G}_m/S)^{\otimes n}, -) \cong t^n \quad (78)$$

and

$$(-)(n)[n] \cong - \otimes M(\mathbb{G}_m/S)^{\otimes n} \cong s^n. \quad (79)$$

Corollary 2.46. Let \mathbb{R} be a cartesian commutative monoid of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$ equipped with a morphism of monoids $\mathbb{T} \rightarrow \mathbb{R}$, for example, $\mathbb{R} = \mathbb{K}$ or $\mathbb{R} = \mathbb{T}$. The 2-functor $H(-, \mathbb{R})$ is a triangulated Sm -motivic category in the sense of [CD12, Def.2.4.45], and a unitary symmetric monoidal stable homotopy 2-functor in the sense of [Ayo07, Def.1.4.1, Def.2.3.1]. Consequently, we have all the six operations and properties as described in Theorem 2.1. See the statement of the theorem for references.

Proof. By definition, a triangulated Sm -motivic category is a triangulated Sm -premotivic category which satisfies Homotopy, Stability, Localisation and the Adjoint Property, the latter being: f_* has a right adjoint for every proper morphism f . We have seen all of this in Propositions 2.34, 2.39, 2.43, 2.40, and 2.38, respectively.

On the other hand, by definition, a stable homotopy 2-functor is a 2-functor H satisfying: $H(\emptyset) = 0$, Properties (1a), (1b), (2) of Def. 2.3, Homotopy, Stability, and Localisation. A stable homotopy 2-functor is unitary symmetric monoidal if it takes values in (unital) symmetric monoidal triangulated categories, cf. Def. 6.4, and satisfies Property (4) of Def. 2.3. \square

Remark 2.47. Ayoub actually asks that $i^*i_* \rightarrow \text{id}$ be an isomorphism for any immersion. Since any quasi-projective immersion factors as a closed and open immersion, it suffices to consider the two cases i is a closed immersion and i is an open immersion. The closed immersion case is part of the Localisation property as we state it. For the case when i is an open immersion, notice that Property (4) of Def. 2.3 implies that $\text{id} \rightarrow i^*i_\#$ is an isomorphism. Then we notice that i^*i_* is right adjoint to $i^*i_\#$, and a left adjoint is an equivalence if and only if its right adjoint is an equivalence.

Notation 2.48. We write

$\mathbb{Z}(n)$ for the complex denoted by $z^n(X, 2n - *)$ in [Blo86]. It is the following complex of presheaves concentrated in cohomological degrees $\leq 2n$. For $X \in \text{QProj}(k)$ the group $\mathbb{Z}(n)(X)^i$ is the free abelian group of codimension n closed irreducible subsets of $X \times \text{Spec}(K(t_0, \dots, t_{2n-i})/1 = \sum t_j)$ whose intersection with each of the faces $t_j = 0$ has pure codimension n inside that face. The differentials are the alternating sums of the intersections with the faces.

$A(n) = \mathbb{Z}(n) \otimes A$ for any abelian group A .

$\text{CH}^n(X, 2n-i)$ is the i th cohomology group of $\mathbb{Z}(n)(X)$.

$\text{CH}^n(X, 2n-i; A)$ is the i th cohomology group of $A(n)(X)$.

One of the most important properties of the complexes $\mathbb{Z}(n)$ is the localisation property.

Theorem 2.49 ([Blo86, p.269]). For any any closed immersion $Z \rightarrow Y$ of codimension d in QProj/k with open complement $U \rightarrow Y$, there exists a canonical quasi-isomorphism of complexes of abelian groups

$$\mathbb{Z}(n-d)(Z) \xrightarrow{q.i.} \text{Cone}\left(\mathbb{Z}(n)(Y) \rightarrow \mathbb{Z}(n)(U)\right)[1]. \quad (80)$$

Corollary 2.50. For any $X \in \text{QProj}/k$, $n, i \in \mathbb{Z}$, and any abelian group A we have

$$H^i(A(n)(X)) \cong \mathbb{H}_{\text{Nis}}^i(X, A(n)) \quad (81)$$

Proof. The localisation sequence implies that the presheaf $\mathbb{Z}(n)$ satisfies Condition (2) of Theorem 2.11. Since $\mathbb{Z}(n)$ is a presheaf of complexes of free abelian groups, we can apply $- \otimes A$ directly, and $A(n)$ also satisfies Condition (2) of Theorem 2.11. Therefore, $A(n)$ also satisfies Condition 1. \square

We will use the following result of Geisser–Levine.

Theorem 2.51 ([GL00, Proposition 3.1, Theorem 8.5]). *For any $X \in \text{Sm}/k$ there are canonical functorial isomorphisms*

$$\text{CH}^n(X, 2n-i; \mathbb{Z}/p) \cong \mathbb{H}_{\text{Nis}}^{i-n}(X, \underline{K}_n^M/p). \quad (82)$$

Proof. In [GL00, Theorem 8.5] Geisser–Levine show that on the small Zariski site of a smooth k -variety X , the complex $\mathbb{Z}/p(n)$ is quasi-isomorphic to the complex concentrated in degree n with the Zariski sheaf v^n in degree n (their statement is a little strange, but $\tau_{\leq n} R\mathcal{E}_* v^n[-n] = v^n[-n]$ due to v^n being an étale sheaf). We don't recall what v^n is because it doesn't matter to us at the moment, we will just observe that Geisser–Levine's [GL00, Proposition 3.1] says that it is isomorphic to the sheafification of \underline{K}_n^M/p (since v^n is an étale sheaf, [GL00, Proposition 3.1] shows that the Zariski, Nisnevich, and étale sheafifications of \underline{K}_n^M/p are all the same).

So we have a quasi-isomorphism

$$\mathbb{Z}/p(n)_{\text{Nis}} \cong \underline{K}_n^M/p[-n] \quad (83)$$

in $C(\text{Shv}_{\text{Nis}}(\text{Sm}/k))$. It remains to observe that the cohomology of $\mathbb{Z}/p(n)$ is the same as its Nisnevich hypercohomology, Corollary 2.50. \square

Corollary 2.52. *For any smooth k -variety $f : X \rightarrow k$, and any commutative \mathbb{Z}/p -algebra A we have canonical functorial isomorphisms*

$$\text{CH}^n(X, 2n-i; A) \cong \text{Hom}_{\text{H}(k, \mathbb{K} \otimes A)}(\mathbb{1}, f_* f^* \mathbb{1}(n)[i]). \quad (84)$$

Proof. Assume for the moment that \mathbb{K} is $\mathcal{S}_{\text{Htp}, \text{Stb}}$ -local. Then we have isomorphisms, [Nee01, Lemma 9.1.5]

$$\text{Hom}_{\text{H}(k, \mathbb{K})}(\mathbb{1}, f_* f^* \mathbb{1}(n)[i]) \quad (85)$$

$$\text{(Adjunction)} \quad \cong \text{Hom}_{\text{H}(X, \mathbb{K})}(\mathbb{1}, \mathbb{1}(n)[i]) \quad (86)$$

$$\text{(Nota.2.45)} \quad \cong \text{Hom}_{\text{H}(X, \mathbb{K})}(t^n \mathbb{K}, \mathbb{K}[i-n]) \quad (87)$$

$$\text{(\mathcal{S}_{Htp, Stb}-locality, Rap. 6.12(3))} \quad \cong \text{Hom}_{D(\mathbb{K}_X\text{-mod})}(t^n \mathbb{K}, \mathbb{K}[i-n]) \quad (88)$$

$$\text{(Equa.45)} \quad \cong \mathbb{H}_{\text{Nis}}^{i-n}(X, \underline{K}_n^M/p) \quad (89)$$

$$\text{(Thm.2.51)} \quad \cong \text{CH}^n(X, 2n-i; \mathbb{Z}/p). \quad (90)$$

Now \mathcal{S}_{Htp} -locality of \mathbb{K} follows from \mathbb{A}^1 -invariance for CH , [Blo86, p.269]. For \mathcal{S}_{Stb} -locality, use the Projective Bundle Theorem, [Blo86, p.269]. \square

Corollary 2.53. *Whenever k is a finite field, or the algebraic closure of a finite field, for any commutative \mathbb{Z}/p -algebra A we have*

$$\text{Hom}_{\text{H}(k, A)}(\mathbb{1}, \mathbb{1}(i)[j]) = \begin{cases} A & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (91)$$

Proof. Since \mathbb{Z}/p is a field, any \mathbb{Z}/p -module is flat and so it suffices to prove the case $A = \mathbb{Z}/p$.

By Theorem 2.51 and Corollary 2.52 it suffices to prove that $\mathbb{H}_{\text{Nis}}^{i-n}(k, \underline{K}_n^M/p) = 0$ unless $n = i = 0$. Fields have Nisnevich cohomological dimension zero, Remark 2.10(1), so we only need to consider the case $n = i$. Since $\mathbb{H}_{\text{Nis}}^0(k, \underline{K}_n^M/p) = \underline{K}_n^M/p(k)$, it suffices to show that the Milnor K -theory has no p -torsion for $n > 0$. For finite fields, this property is due to Steinberg, [Mil70, Exam.1.5], and for the algebraic closures, it follows from the fact that Milnor K -theory commutes with filtered colimits. \square

Proposition 2.54 (Projective bundle formula). *Let A be a commutative \mathbb{Z}/p -algebra, and let $E \rightarrow X$ be a vector bundle of dimension n in \mathbf{QProj}/k , and $p : \mathbb{P}(E) \rightarrow X$ its associated projective bundle. Then we have the following isomorphisms in $H(X, A)$*

$$p_* \mathbb{1}_E \cong \bigoplus_{i=0}^{n-1} \mathbb{1}(-i)[-2i], \quad M(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{n-1} \mathbb{1}(i)[2i].$$

Proof. For easy of notation, set $P = \mathbb{P}(E)$.

Applying $\mathcal{H}om(-, \mathbb{1})$ to the right isomorphism gives the left one, so it suffices to prove the right one (note that $M(P) \cong p_{\#} p^* \mathbb{1}_X$ and $\mathcal{H}om(p_{\#} p^* \mathbb{1}, \mathbb{1}) \cong \mathcal{H}om(\mathbb{1}, p_* p^* \mathbb{1})$). Now the right isomorphism in $H(X, A)$ is the image of this isomorphism in $H(X, \mathbb{Z}/p)$ under the canonical functor $H(X, \mathbb{Z}/p) \rightarrow H(X, A)$, so it suffices to treat the case $A = \mathbb{Z}/p$.

Now the standard proof applies, cf. [Voe00b, Prop.3.5.1], [Voe96, Thm.4.2.7]. Let us sketch it. For any $Y \in \mathbf{Sm}/X$, we have a canonical morphism

$$\begin{aligned} \text{Pic}(Y) &\rightarrow \mathbb{H}_{\text{Nis}}^1(Y, \mathbb{G}_m) \rightarrow \mathbb{H}_{\text{Nis}}^1(Y, \underline{K}_1^M/p) \rightarrow \text{Hom}_{D(\mathbb{K}_X\text{-mod})}(\iota \mathbb{K}(Y), \mathbb{K}[1]) \\ &\rightarrow \text{Hom}_{H(X, \mathbb{K})}(\iota \mathbb{K}(Y), \mathbb{K}[1]) \cong \text{Hom}_{H(X, \mathbb{K})}(M(Y), \mathbb{1}(1)[2]) \end{aligned}$$

Note that this is functorial in Y . The class of $\mathcal{O}(1)$ in $\text{Pic}(P)$ defines a morphism $\tau_1 : M(P) \rightarrow \mathbb{1}(1)[2]$, and for $n \geq 0$ we define τ_n and Σ as the composition and sum

$$\tau_n : M(P) \xrightarrow{\text{diag.}} M(P^n) \cong M(P)^{\otimes n} \xrightarrow{\tau^{\otimes n}} \mathbb{1}(n)[2n], \quad \Sigma : M(P) \xrightarrow{\Sigma \tau_1} \bigoplus_{i=0}^{n-1} \mathbb{1}(i)[2i]. \quad (92)$$

The claim is that Σ is an isomorphism. Using Mayer-Vietoris and functoriality of τ , we can assume that E is a trivial vector bundle. Then one uses induction on n , and the Mayer-Vietoris triangle

$$M(\mathbb{A}^n - \{0\}) \rightarrow M(\mathbb{P}^n - \{0\}) \oplus M(\mathbb{A}^n) \rightarrow M(\mathbb{P}^n) \rightarrow M(\mathbb{A}^n - \{0\})[1]. \quad (93)$$

By \mathbb{A}^1 -invariance, we have isomorphisms $M(\mathbb{A}_X^n) \cong M(X)$ and $M(\mathbb{P}^n - \{0\}) \cong M(\mathbb{P}^{n-1})$. So by induction, the second term in our triangle is $(\bigoplus_{i=0}^{n-2} \mathbb{1}(i)[2i]) \oplus \mathbb{1}$. So it suffices to identify the fourth term $M(\mathbb{A}^n - \{0\})[1]$ as $\mathbb{Z}(n-1)[2n-2] \oplus \mathbb{1}$ in a way that forms a morphism of distinguished triangles. This is done by constructing an explicit class in $\mathbb{H}^{n-1}(\mathbb{A}^n - \{0\}, \mathbb{G}_m^{\otimes n})$ inducing an appropriate morphism $M(\mathbb{A}^n - \{0\})[1] \rightarrow \mathbb{Z}(n-1)[2n-2]$. See the proof of [Voe96, Thm.4.2.7] for more details. \square

Remark 2.55. Many of the other proofs from [Voe00b] also work in our setting. See [Voe00b, Section 2.2] for a list. Note that in our setting, $f_*f^!\mathbb{1}_S$ plays the rôle of $M^c(X)$ for $f : X \rightarrow S$ in $Sm(S)$, $S \in \text{QProj}(k)$.

3 Mixed Stratified Tate Motives

In the following we always assume that our varieties are defined over $\overline{\mathbb{F}}_p$ and that \mathbb{k} is a field of characteristic p . We will often drop the \mathbb{k} from the notation. In this section we define the category of stratified mixed Tate motives as a full subcategory of $H(X, \mathbb{K} \otimes \mathbb{k}) = H(X, \mathbb{k}) = H(X)$ as constructed in the last section. We will then consider a weight structure on this category, prove a formality result, and state the Erweiterungssatz.

3.1 Stratified mixed Tate motives

Let (X, \mathcal{S}) be an affinely stratified variety over $\overline{\mathbb{F}}_p$, i.e. a variety X with a finite partition into locally closed subvarieties (called the strata of X)

$$X = \bigcup_{s \in \mathcal{S}} X_s, \quad (94)$$

such that each stratum X_s is isomorphic to \mathbb{A}^n for some n , and the closure \overline{X}_s is a union of strata. The embeddings are denoted by $j_s : X_s \hookrightarrow X$. The prime example we always have in mind here is the flag variety of a reductive group with its Bruhat stratification. Starting from this datum, [SW16] defines the category of *stratified mixed Tate motives* on X , which we recall in this paragraph. We start with the basic case of just one stratum.

Definition 3.1. For $X \cong \mathbb{A}^n$, denote by $\text{MTDer}(X, \mathbb{k}) = \text{MTDer}(X)$ the full triangulated subcategory of $H(X, \mathbb{k})$ generated by motives isomorphic to $\mathbb{1}_X(p)$ for $p \in \mathbb{Z}$. Recall that by $\mathbb{1}_X$ we denote the tensor unit in $H(X, \mathbb{k})$.

We shall make extensive use of the following statement.

Proposition 3.2. *For $X \cong \mathbb{A}^n$, we have the following equivalence of monoidal \mathbb{k} -linear categories:*

$$\text{MTDer}(X) \cong \mathbb{k}\text{-mod}^{\mathbb{Z} \times \mathbb{Z}} \cong \text{Der}^b(\mathbb{k}\text{-mod}^{\mathbb{Z}}). \quad (95)$$

Here, the $\mathbb{k}\text{-mod}^{\mathbb{Z} \times \mathbb{Z}}$ denotes the category of bigraded, finite dimensional vector spaces over \mathbb{k} and $\text{Der}^b(\mathbb{k}\text{-mod}^{\mathbb{Z}})$ is the bounded derived category of graded, finite dimensional vector spaces over \mathbb{k} .

We choose the isomorphisms such that $\mathbb{L}_X(i)[j]$ corresponds to \mathbb{k} sitting in degree (i, j) in $\mathbb{k}\text{-mod}^{\mathbb{Z} \times \mathbb{Z}}$ and \mathbb{k} sitting in degree i with respect to the grading and cohomological degree $-j$ in $\text{Der}^b(\mathbb{k}\text{-mod}^{\mathbb{Z}})$.

Note that this equips $\text{MTDer}(X)$ with a natural t -structure. We denote the j -th cohomology functor by

$$\mathcal{H}^j : \text{MTDer}(X) \rightarrow \mathbb{k}\text{-mod}^{\mathbb{Z}} \quad (96)$$

and by $\text{MTDer}(X)^{\leq n}$, resp. $\text{MTDer}(X)^{\leq n}$, the full subcategories of objects $E \in \text{MTDer}(X)$ with $\mathcal{H}^j(E) = 0$ for all $j > n$, resp. $j < n$.

Proof. Follows from Corollary 2.53. \square

We can now proceed to the general case. Since our category should be closed under taking Verdier duals and other reasonable combinations of the six functors, we have to assume that (X, \mathcal{S}) fulfils an additional condition:

Definition 3.3. (X, \mathcal{S}) is called *Whitney-Tate* if and only if for all $s, t \in \mathcal{S}$ and $M \in \text{MTDer}(X_s)$ we have $j_t^* j_{s*} M \in \text{MTDer}(X_t)$.

From now on we always assume that (X, \mathcal{S}) is Whitney-Tate. In [SW16] it is shown that (partial) flag varieties and other examples are indeed Whitney-Tate.⁴

Definition 3.4. The category of *stratified mixed Tate motives* on X , denoted by $\text{MTDer}_{\mathcal{S}}(X, \mathbb{k}) = \text{MTDer}_{\mathcal{S}}(X)$, is the full subcategory of $\text{H}(X)$ consisting of objects M such that $j_s^* M \in \text{MTDer}(X_s)$ for all $s \in \mathcal{S}$.

Remark 3.5. Because we assumed X to be Whitney-Tate, we could have also required the equivalent condition $j_s^! M \in \text{MTDer}(X_s)$ for all $s \in \mathcal{S}$, see [SW16, Remark 4.6]

3.2 Affinely stratified maps

The right definition of a map between affinely stratified varieties is different to the usual definition of a stratified map, as defined for example in [GM88].

Definition 3.6. Let (X, \mathcal{S}) and (Y, \mathcal{S}') be affinely stratified varieties. We call $f : X \rightarrow Y$ an *affinely stratified map* if

1. for all $s \in \mathcal{S}'$ the inverse image $f^{-1}(Y_s)$ is a union of strata;
2. for each X_s mapping into $Y_{s'}$, the induced map $f : X_s \rightarrow Y_{s'}$ is a projection $\mathbb{A}^n \times \mathbb{A}^m \rightarrow \mathbb{A}^m$.

⁴ In Proposition A.2. [SW16] gives a sufficient condition for a stratified scheme to be Whitney-Tate, involving the existence of certain resolutions of singularities of the closure of strata. In one step of the proof they use absolute purity, which is not proven for our formalism. But here relative purity actually suffices, and the proposition still applies in our setting.

We now prove the compatibility of stratified mixed Tate motives with pullback and pushforward functors coming from affinely stratified maps.

Lemma 3.7. *Let $X \in \text{Sch}(k)$ be smooth. Consider*

$$s : X \rightrightarrows \mathbb{A}_X^n : p \quad (97)$$

where p denotes the projection and s the zero section. Then

$$\begin{aligned} p_*(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X & p_!(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X(-n)[-2n] \\ p^*(\mathbb{1}_X) &= \mathbb{1}_{\mathbb{A}_X^n} & p^!(\mathbb{1}_X) &= \mathbb{1}_{\mathbb{A}_X^n}(n)[2n] \\ s^*(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X & s^!(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X(-n)[-2n] \end{aligned}$$

Furthermore $D_X(\mathbb{1}_X(m)[2m]) = \mathbb{1}_X(\dim X - m)[2\dim X - 2m]$ where

$$D_X = \mathcal{H}om_X(-, f^!(\mathbb{1}))$$

for $f : X \rightarrow k$ is the structural morphism, cf. Equation 10.

Proposition 3.8. *Let (X, S) and (Y, S') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ a affinely stratified map. Then the induced functors restrict to stratified mixed Tate motives on X and Y . In formulas*

$$f_*, f_! : \text{MTDer}_S(X) \rightrightarrows \text{MTDer}_{S'}(Y) : f^*, f^! \quad (98)$$

Also the internal Hom, duality and tensor product restrict.

Proof. Duality preserves stratified mixed Tate motives because (X, S) and (Y, S') are Whitney-Tate; this follows from Remark 3.5. So we only have to prove the statements for half of the six functors, cf. Theorem 2.1(12). The statement for f^* follows directly from the definitions.

We consider $f_!$ next. Let $E \in \text{MTDer}_S(X)$. We have to show $v^* f_! E \in \text{MTDer}(Y_s)$ for all strata $v : Y_s \hookrightarrow Y$. By base change applied to the cartesian diagram

$$\begin{array}{ccc} f^{-1}(Y_s) & \xrightarrow{w} & X \\ g \downarrow & & \downarrow f \\ Y_s & \xrightarrow{v} & Y \end{array}$$

we have to show that $g_! w^* E \in \text{MTDer}(Y_s)$. This can be done by an induction on the number of strata in $f^{-1}(Y_s)$. Denote by j the inclusion of an open stratum X_s in $f^{-1}(Y_s)$ and by i the one of the complement. We obtain the distinguished triangle

$$g_! j_! j^* w^* E \longrightarrow g_! w^* E \longrightarrow g_! i_! i^* w^* E \xrightarrow{+1}$$

Let us first consider the left hand side. By assumption we have $j^* w^* E = j_s^* E \in \text{MTDer}(X_s)$. Since f is a affinely stratified map, $g j$ is a projection $\mathbb{A}^n \times \mathbb{A}^m \rightarrow \mathbb{A}^n$ and

by Lemma 3.7 $(gj)!$ maps $\mathrm{MTDer}(X_s)$ to $\mathrm{MTDer}(Y_s)$. Hence we have $g_! j_! j^* w^* E \in \mathrm{MTDer}_{S'}(Y)$. The right hand side is a stratified mixed Tate motive by induction. Now the statement follows from the fact that $\mathrm{MTDer}(Y_s)$ is closed under extensions.

The statement for the tensor product follows immediately, since pullback is a tensor functor, and we are done. \square

3.3 Weights

Weight structures—as first considered in [Bon10]—provide a very concise framework for the powerful *yoga of weights*, as applied, for example, in the proof of the Weil conjectures or the decomposition theorem for perverse sheaves.

Definition 3.9. Let \mathcal{C} be a triangulated category. A *weight structure* on \mathcal{C} is a pair $(\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ of full subcategories of \mathcal{C} such that with $\mathcal{C}_{w \leq n} := \mathcal{C}_{w \leq 0}[n]$ and $\mathcal{C}_{w \geq n} := \mathcal{C}_{w \geq 0}[n]$ the following conditions are satisfied:

1. $\mathcal{C}_{w \leq 0}$ and $\mathcal{C}_{w \geq 0}$ are closed under direct summands;
2. $\mathcal{C}_{w \leq 0} \subseteq \mathcal{C}_{w \leq 1}$ and $\mathcal{C}_{w \geq 1} \subseteq \mathcal{C}_{w \geq 0}$;
3. for all $X \in \mathcal{C}_{w \leq 0}$ and $Y \in \mathcal{C}_{w \geq 1}$, we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
4. for any $X \in \mathcal{C}$ there is a distinguished triangle $A \longrightarrow X \longrightarrow B \xrightarrow{+1}$ with $A \in \mathcal{C}_{w \leq 0}$ and $B \in \mathcal{C}_{w \geq 1}$

The full subcategory $\mathcal{C}_{w=0} = \mathcal{C}_{w \leq 0} \cap \mathcal{C}_{w \geq 0}$ is called the *heart of the weight structure*.

Unlike in the setting of motives with rational coefficients as considered in [SW16], it is not a priori known if our category of motives can be equipped with a weight structure (the standard proofs rely on the existence of some kind of resolution of singularities and do not work for torsion coefficients equal to the characteristic of the base). Nevertheless we can define a weight structure directly on the category of stratified mixed Tate motives and also prove compatibilities with the six functors (at least for affinely stratified maps).

We start by defining weight structures for the Tate motives on the affine strata. Here we want $\mathbb{1}_{\mathbb{A}^n}(p)[q]$ to have weight $q - 2p$.

Definition 3.10. Let $\mathrm{MTDer}(\mathbb{A}^n)_{w \leq 0}$ (resp. $\mathrm{MTDer}(\mathbb{A}^n)_{w \geq 0}$) be the full subcategory of $\mathrm{MTDer}(\mathbb{A}^n)$ consisting of objects isomorphic to finite direct sums of $\mathbb{1}_{\mathbb{A}^n}(p)[q]$ for $q \leq 2p$ ($q \geq 2p$). This defines a weight structure on $\mathrm{MTDer}(\mathbb{A}^n)$.

Proof. We use Proposition 3.2 to identify $\mathrm{MTDer}(\mathbb{A}^n)$ with the derived category of graded vector spaces. Here the axioms of a weight structure are easily checked. \square

We can now obtain a weight structure for stratified mixed Tate motives by gluing.

Definition 3.11. Let (X, S) be an affinely Whitney-Tate stratified variety. Then we obtain a weight structure on $\mathrm{MTDer}_S(X)$ by setting

$$\begin{aligned}\mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq 0} &:= \{M \mid j_s^* M \in \mathrm{MTDer}(X_s)_{w \leq 0} \text{ for all } s \in \mathcal{S}\} \\ \mathrm{MTDer}_{\mathcal{S}}(X)_{w \geq 0} &:= \{M \mid j_s^! M \in \mathrm{MTDer}(X_s)_{w \geq 0} \text{ for all } s \in \mathcal{S}\}\end{aligned}$$

Proof. [SW16, Proposition 5.1]. \square

With this definition we have the following compatibilities with the six functors.

Proposition 3.12. *Let (X, \mathcal{S}) and (Y, \mathcal{S}') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ an affinely stratified map. Then*

1. *the functors $f^*, f_!$ are weight left exact, i.e. they preserve $w \leq 0$;*
2. *the functors $f^!, f_*$ are weight right exact, i.e. they preserve $w \geq 0$;*
3. *the tensor product is weight left exact, i.e. restricts to*

$$\mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq n} \times \mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq m} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq n+m} \quad (99)$$

4. *Verdier duality reverses weights, i.e. restricts to*

$$D_X : \mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq n}^{\mathrm{op}} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)_{w \geq -n} \quad (100)$$

5. *the internal Hom functor $\mathcal{H}om_X$ is weight right exact, i.e. restricts to*

$$\mathrm{MTDer}_{\mathcal{S}}(X)_{w \leq n}^{\mathrm{op}} \times \mathrm{MTDer}_{\mathcal{S}}(X)_{w \geq m} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)_{w \geq m-n} \quad (101)$$

6. *For f smooth $f^!$ and f^* are weight exact;*
7. *For f proper $f_!$ and f_* are weight exact;*
8. *If X is smooth $\mathbb{1}_X(n)[2n]$ is of weight zero for all $n \in \mathbb{Z}$.*

Proof. Follows by the same arguments as in Proposition 3.8 while using Lemma 3.7 to see that the pullbacks and pushforwards associated to projections $p : \mathbb{A}^n \times \mathbb{A}^m \rightarrow \mathbb{A}^n$ preserve weights. \square

3.4 Pointwise purity, formality and tilting

In this section we use *pointwise purity* to prove that our category of stratified mixed Tate motives can be realised as the homotopy category of motives of weight zero. This is the analogue of [SW16, Theorem 9.2]. It can also be restated as a *formality* result: The category of stratified mixed Tate motives is equivalent to a dg-derived category over a formal dg-algebra. In our applications weight zero motives will turn out to exactly be direct sums of (appropriately shifted and twisted) parity motives.

Definition 3.13. Let $? \in \{*, !\}$. A motive $M \in \mathrm{MTDer}_{\mathcal{S}}(X)$ is called *pointwise $?$ -pure* if for all $s \in \mathcal{S}$

$$i_s^? M \in \mathrm{MTDer}(X_s)_{w=0}. \quad (102)$$

If both conditions are satisfied, the motive is called *pointwise pure*.

We list some compatibilities of the six functors with pointwise purity.

Proposition 3.14. *Let (X, \mathcal{S}) and (Y, \mathcal{S}') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ an affinely stratified map. Then*

1. *For f smooth $f^!$ and f^* preserve pointwise purity;*
2. *For f proper $f_!$ and f_* preserve pointwise purity.*

Proof. (1) follows from $f^! = f^*(-d)[-2d]$, where d is the relative dimension of f .
 (2) follows from base change and $f_! = f_*$. \square

The following Lemma is crucial to all following arguments, and depends on the fact that there are no non-trivial extension between the Tate objects on $\overline{\mathbb{F}}_q$:

Lemma 3.15. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety. Let $E, F \in \text{MTDer}_{\mathcal{S}}(X)$ such that E is pointwise $*$ -pure of weight zero and F is pointwise $!$ -pure of weight zero. Then*

$$\text{Hom}_{\mathbf{H}(X)}(E, F[a]) = 0. \quad (103)$$

for all $a \neq 0$.

Proof. See [SW16, Corollary 6.3]. We repeat a proof here. We proceed by induction on the number of strata. Denote by $j : \mathbb{A}^n = U \hookrightarrow X$ the inclusion of an open stratum in X and by $i : Z \hookrightarrow X$ its closed complement. Hence there is an distinguished triangle

$$j_! j^* E \longrightarrow E \longrightarrow i_! i^* E \xrightarrow{+1}$$

and $\text{Hom}_{\mathbf{H}(X)}(E, F[a])$ fits in an exact sequence

$$\text{Hom}_{\mathbf{H}(\mathbb{A}^n)}(j^* E, j^! F[a]) \longrightarrow \text{Hom}_{\mathbf{H}(X)}(E, F[a]) \longrightarrow \text{Hom}_{\mathbf{H}(Z)}(i^* E, i^! F[a])$$

where the left term vanishes using Proposition 3.2 (this is where we use that there are non-trivial extension between the Tate objects) and the right hand term vanishes by induction. The statement follows. \square

Theorem 3.16 (Tilting). *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety, such that all objects of $\text{MTDer}_{\mathcal{S}}(X)_{w=0}$ are additionally pointwise pure. Then there is an equivalence of categories, called tilting,*

$$\Delta : \text{MTDer}_{\mathcal{S}}(X) \xrightarrow{\sim} \text{Hot}(\text{MTDer}_{\mathcal{S}}(X)_{w=0}). \quad (104)$$

Proof. By our construction $\text{MTDer}_{(\mathcal{S})}(X)$ is a full subcategory of a derived category of a Grothendieck abelian category, namely $\mathcal{A} = \mathbb{R}\text{-mod} = (\mathbb{K}_X \otimes \mathbb{k})\text{-mod}$ (see Remark 6.13). For every weight zero object, considered as an object in $D(\mathbb{R}\text{-mod})$, choose a representative in $C(\mathbb{R}\text{-mod})$ which is both fibrant and cofibrant, cf. Section 6.3. The collection \mathcal{T} of these choices of representatives fulfills the following properties:

1. For all $E, F \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathrm{Hot}(\mathcal{A})}(E, F[n]) = \mathrm{Hom}_{\mathrm{Der}(\mathcal{A})}(E, F[n])$, since they are both fibrant and cofibrant.
2. For all $E, F \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathrm{Der}(\mathcal{A})}(E, F[n]) = 0$ for all $n \neq 0$, by the pointwise purity assumption and Lemma 3.15.
3. \mathcal{T} generates $\mathrm{MTDer}_{\mathcal{S}}(X)$ as a triangulated category, because using an induction on the number of strata and the weight structure.

A collection fulfilling those properties is often called a *tilting collection* in the literature. The statement now follows from [Ric89, Proposition 10.1] or [Kel93, Theorem 1], see [SW16, Appendix B] for a nice sketch of the proof. \square

Theorem 3.17 (Formality). *The last theorem can also be stated as natural equivalence*

$$\mathrm{MTDer}_{(\mathcal{S})}(X) \xrightarrow{\sim} \mathrm{dgDer}\text{-}(E, d=0) \quad (105)$$

where the right hand side denotes the dg-derived category of the formal (equipped with a trivial differential) graded dg-algebra

$$E = \bigoplus_{i,j \in \mathbb{Z}} \mathrm{Hom}_{H(X)}(L, L(i)[j]) \quad (106)$$

where L is a direct sum of objects L_i generating $\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$ with respect to direct sum, shifts, $(n)[2n]$, and isomorphism.

3.5 Erweiterungssatz

The *Erweiterungssatz* as first stated in [Soe90] and reproven in a more general setting in [Gin91] allows a *combinatorial* description of pointwise pure weight zero sheaves on X in terms of certain modules over the cohomology ring of X . In the case of X being the flag variety, these modules are called *Soergel modules*.

In our setting, the same results hold by replacing the usual singular or étale cohomology ring by the motivic one.

Definition 3.18. Let $p : X \rightarrow \mathrm{Spec}(k) \in \mathrm{QProj}/k$ and $E \in H(X)$. Denote by

$$\mathbb{H}(E) := \bigoplus_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \mathrm{Hom}_{H(X)}(\mathbb{1}_X, E(i)[j]) \quad (107)$$

the *hypercohomology* functor. Note that $\mathbb{H}(E)$ is naturally a bigraded right module over $\mathbb{H}(X, \mathbb{k}) := \mathbb{H}(X) := \mathbb{H}(\mathbb{1}_X) = H_{\mathcal{M}}^{\bullet}(X, \mathbb{k}(\bullet))$, the motivic cohomology ring of X .

Motivic cohomology (at least for smooth schemes) can be identified with higher Chow groups, which are usually quite infeasible for computation. For affinely stratified varieties everything gets much easier:

Theorem 3.19. *Let (X, \mathcal{S}) be an affinely stratified and irreducible variety of dimension n over k . Then:*

1. *The motivic cohomology ring $\mathbb{H}(X) = H_{\mathcal{M}}^{\bullet}(X, \mathbb{k}(\bullet))$ is concentrated in degrees $(2i, i)$ and we have*

$$H_{\mathcal{M}}^{2i}(X, \mathbb{k}(i)) = \bigoplus_{\substack{s \in \mathcal{S}, \\ \dim X_s = i}} \mathbb{k}. \quad (108)$$

2. *If X is furthermore smooth, there are \mathbb{k} -algebra isomorphisms*

$$\mathbb{H}(X) \cong CH^{\bullet}(X, \mathbb{k}) \cong CH^{\bullet}(X, \mathbb{Z}) \otimes \mathbb{k} \quad (109)$$

where for a ring Λ we denote by $CH^{\bullet}(X, \Lambda) = CH_{n-\bullet}(X, \Lambda)$ the classical Chow ring of X with coefficients in Λ .

Proof. (1) follows by a standard argument using an induction on the number of strata, which we quickly repeat here. Denote by $j : \mathbb{A}^n = U \hookrightarrow X$ the inclusion of an open stratum in X and by $i : Z \hookrightarrow X$ its closed complement. Let $p : X \rightarrow \operatorname{Spec}(k)$ be the structure map. Then by the localization property we have the distinguished triangle in $\operatorname{Der}^b(\mathbb{k}\text{-mod})$

$$p_* i_* i^! \mathbb{1}_X \longrightarrow p_* \mathbb{1}_X \longrightarrow p_* j_* j^* \mathbb{1}_X \xrightarrow{+1} \quad (110)$$

For the right hand side we have

$$p_* j_* j^* \mathbb{1}_X = p_* j_* \mathbb{1}_U = \mathbb{1}_{\operatorname{Spec}(k)}. \quad (111)$$

Denote by d the codimension of Z in X , then by relative purity we have

$$p_* i_* i^! \mathbb{1}_X = p_* i_* \mathbb{1}_Z(-d)[-2d] \quad (112)$$

for the left hand side and its cohomology is (up to the shift / twist) the motivic cohomology of Z , for which we can apply the induction hypothesis. In particular, the left hand side $p_* i_* i^! \mathbb{1}_X$ is concentrated in even cohomological degrees and the statement follows by the cohomology long exact sequence associated to the distinguished triangle together with Lemma 3.2.

(2) The first equality follows from (1) and Corollary 2.52 (here we need to assume that X is smooth). The second equality follows since the Chow groups of an affinely stratified variety are indeed free. One can see this by using the same arguments as in (1) and the localization property for higher Chow groups, see [Blo86] and [Lev94]. \square

So in our setting the Erweiterungssatz reads:

Theorem 3.20 (Erweiterungssatz). *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified and proper variety and $E, F \in \operatorname{MTDer}_{\mathcal{S}}(X)$ pointwise pure. Assume additionally that for each embedding j of a stratum $\mathbb{H}E \rightarrow \mathbb{H}j_* j^* E$ is surjective and $\mathbb{H}j_! j^! F \rightarrow \mathbb{H}F$ is injective. Then hypercohomology induces an isomorphism*

$$\mathrm{Hom}_{\mathbb{H}(X)}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{H}(X)}^{\mathbb{Z} \times \mathbb{Z}}(\mathbb{H}(E), \mathbb{H}(F)) \quad (113)$$

where the right hand denotes morphisms of bigraded $\mathbb{H}(X)$ -modules.

Proof. [SW16, Theorem 8.4]. \square

Corollary 3.21. *Let (X, S) be an affinely Whitney-Tate stratified and proper variety and assume that all objects in $\mathrm{MTDer}_S(X)_{w=0}$ satisfy the conditions of Theorem 3.20. Then hypercohomology induces a fully faithful embedding*

$$\mathbb{H} : \mathrm{MTDer}_S(X)_{w=0} \rightarrow \mathrm{mod}^{\mathbb{Z} \times \mathbb{Z}}\text{-}\mathbb{H}(X) \quad (114)$$

of weight zero motives into the category of bigraded $\mathbb{H}(X)$ -modules. We denote the essential image by $\mathrm{Smod}^{\mathbb{Z} \times \mathbb{Z}}\text{-}\mathbb{H}(X)$.

Remark 3.22. For X affinely stratified, $\mathbb{H}(X)$ is concentrated in degrees $(2i, i)$ by Theorem 3.19. By similar arguments the hypercohomology of all pointwise pure weight zero motives will also only live in degrees $(2i, i)$, hence in fact we have a fully faithful embedding

$$\mathbb{H} : \mathrm{MTDer}_S(X)_{w=0} \rightarrow \mathrm{mod}^{\mathbb{Z}}\text{-}\mathbb{H}(X) \quad (115)$$

into the category of \mathbb{Z} -graded modules, with respect to the diagonal grading.

4 Parity motives, perverse motives and the flag variety

In this section we want to study certain interesting subcategories of our category of stratified mixed Tate motives. The general principle here is that everything works as one is used to from constructible étale sheaves or mixed Hodge modules. As in the last section, all varieties are over $\overline{\mathbb{F}}_p$ and \mathbb{k} is an arbitrary field of characteristic p , which we will often drop from the notation.

4.1 Parity motives

Parity sheaves were first used by [Soe00] as a substitute for intersection complexes in a setting where the decomposition theorem (c.f. [BBD82], [Sai89], [dCM09]) does not hold in general, namely for constructible sheaves with modular coefficients. Then [JMW14] properly axiomatized and classified them. In this section we want to recall their properties and argue that the whole theory works fine in the setting of motives. All one really needs is a six functor formalism, as already stated in the introduction of [JMW14]. We want to remark that our situation is simpler, since we just consider affine strata and hence only trivial local systems.

Definition 4.1. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety. A motive $E \in \text{MTDer}_{\mathcal{S}}(X)$ is called *parity* if it can be decomposed into a direct sum $E_1 \oplus E_2$ such that for all $s \in \mathcal{S}$ and $? \in \{!, *\}$ we have

$$\mathcal{H}^i(j_s^? E_k) = 0 \text{ if } i \not\equiv k \pmod{2}.$$

We denote the full subcategory of motives which are parity by

$$\text{Par}_{\mathcal{S}}(X, \mathbb{k}) = \text{Par}_{\mathcal{S}}(X) \subseteq \text{MTDer}_{\mathcal{S}}(X). \quad (116)$$

As opposed to intersection complexes, the existence of parity sheaves/motives with prescribed support is not known in general. Only their uniqueness.

Theorem 4.2. *For all $s \in \mathcal{S}$ there exists (up to isomorphism) at most one indecomposable parity motive E_s supported on \overline{X}_s with $j_s^* E_s = \mathbb{1}_{X_s}$.*

Proof. [JMW14, Theorem 2.12] translates unchanged. \square

The following theorem is a useful tool for constructing parity sheaves. It can be thought of as an analogue of the decomposition theorem.

Proposition 4.3. *Let (X, \mathcal{S}) and (Y, \mathcal{S}') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ a proper affinely stratified map. Then $f_! = f_*$ preserves the parity condition.*

Proof. This is proven in [JMW14, Prop. 2.34], their condition of f being even is not needed/trivially fulfilled in our setting. For us an even simpler proof, analogous to the one of Proposition 3.8, applies. \square

Under the condition that the closures of all strata admit proper resolutions we can identify the additive category of finite direct sums of (appropriately shifted and twisted) parity motives with the category of weight zero motives.

Theorem 4.4. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that for every $s \in \mathcal{S}$ there exists a proper, affinely stratified map*

$$\pi : \widetilde{X}_s \rightarrow \overline{X}_s \subset X \quad (117)$$

with \widetilde{X}_s smooth, inducing an isomorphism over X_s . Then for every $s \in \mathcal{S}$ there exists an indecomposable, pointwise pure parity motive $E_s \in \text{Par}_{\mathcal{S}}(X)$ with $j_s^ E_s = \mathbb{1}_{X_s}$. The objects in $\text{Par}_{\mathcal{S}}(X)_{w=0}$ are the motives isomorphic to finite direct sums of the $E_s(n)[2n]$ for $n \in \mathbb{Z}$, $s \in \mathcal{S}$ and we have*

$$\text{MTDer}_{\mathcal{S}}(X)_{w=0} = \text{Par}_{\mathcal{S}}(X)_{w=0}. \quad (118)$$

Furthermore every parity motive is a direct sum of motives in $\text{Par}_{\mathcal{S}}(X)_{w=0}$.

Proof. Since \tilde{X}_s is smooth, the dual of $\mathbb{L}_{\tilde{X}_s}$ is $D_{\tilde{X}_s} \mathbb{L}_{\tilde{X}_s} = \mathbb{L}_{\tilde{X}_s}(\dim X_s)[2 \dim X_s]$. Hence the restriction of $\mathbb{L}_{\tilde{X}_s}$ to all strata of \tilde{X}_s using $!$ or $*$ is pure of weight zero and concentrated in even cohomological degrees. Hence $\mathbb{L}_{\tilde{X}_s}$ is parity and pointwise pure of weight zero. Since π is proper, $\pi_! \mathbb{L}_{\tilde{X}_s}$ is also parity and pointwise pure of weight zero by Propositions 3.14 and 4.3. Furthermore by base change $j_s^* \pi_! \mathbb{L}_{\tilde{X}_s} = \mathbb{L}_{X_s}$. So we choose E_s to be the unique indecomposable direct summand of $\pi_! \mathbb{L}_{\tilde{X}_s}$ with $j_s^* E_s = \mathbb{L}_{X_s}$. By Theorem 4.2 we know that these are all weight zero indecomposable parity sheaves—up to shifting and twisting by $(n)[2n]$.

The other statements can be proven along the lines of [SW16, Corollary 6.7]. By a standard induction argument one sees that the $E_s(n)[2n]$ generate $\mathrm{MTDer}_{\mathcal{S}}(X)$ as a triangulated category. The pointwise purity and Proposition 3.14 imply that $\mathrm{Hom}_{\mathrm{H}(X)}(E, F[a]) = 0$ for all $a > 0$ and $E, F \in \mathrm{Par}_{\mathcal{S}}(X)_{w=0}$. By [Bon14, Proposition 1.7(6)] it follows that

$$\mathrm{Par}_{\mathcal{S}}(X)_{w=0} = \mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}. \quad \square$$

The pointwise purity of the indecomposable parity sheaves allows us to apply the tilting result from the last section and we obtain:

Corollary 4.5. *Under the assumptions of Theorem 4.4 there is an equivalence of categories*

$$\mathrm{MTDer}_{\mathcal{S}}(X) \cong \mathrm{Hot}^b(\mathrm{Par}_{\mathcal{S}}(X)_{w=0}). \quad (119)$$

4.2 Perverse motives

The whole theory of perverse sheaves from [BBD82, §1, §2] applies in our setting. Again, all one needs is a six functor formalism. In particular, we can perversely glue the standard t -structures on the categories of mixed Tate motives on the strata—recall that they are just derived categories of graded vector spaces—to obtain a perverse t -structure on the category of stratified mixed Tate motives on an affinely stratified variety. Our goal is to show that—under a technical assumption—the category of stratified mixed Tate motives can be realized as the derived category of perverse motives, the homotopy category of projective perverse motives, or the homotopy category of tilting perverse motives, respectively.

Definition 4.6. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety. Then we obtain a t -structure, called the *perverse t -structure*, on $\mathrm{MTDer}_{\mathcal{S}}(X)$ by setting

$$\begin{aligned} \mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq 0} &:= \left\{ M \mid j_s^* M \in \mathrm{MTDer}(X_s)^{\leq -\dim X_s} \text{ for all } s \in \mathcal{S} \right\}, \\ \mathrm{MTDer}_{\mathcal{S}}(X)^{p \geq 0} &:= \left\{ M \mid j_s^! M \in \mathrm{MTDer}(X_s)^{\geq -\dim X_s} \text{ for all } s \in \mathcal{S} \right\}. \end{aligned}$$

We denote the heart of this t -structure by

$$\mathrm{Per}_{\mathcal{S}}(X, \mathbb{k}) = \mathrm{Per}_{\mathcal{S}}(X) \quad (120)$$

and call its objects *perverse motives* on X .

Proposition 4.7. *Let (X, \mathcal{S}) be affinely Whitney-Tate stratified varieties and $j : W \rightarrow X$ be an inclusion of a union of strata. Then*

1. *the functors $j^*, j_!$ are right t -exact, i.e. they preserve $p \leq 0$;*
2. *the functors $j^!, j_*$ are left t -exact, i.e. they preserve $p \geq 0$;*
3. *the tensor product is weight left exact, i.e. restricts to*

$$\mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq n} \times \mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq m} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq n+m} \quad (121)$$

4. *Verdier duality reverses the t -structure, i.e. restricts to*

$$\mathrm{D}_X : \mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq n, \mathrm{op}} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)^{p \geq -n} \quad (122)$$

5. *the internal Hom functor $\mathcal{H}om_X$ is weight right t -exact, i.e. restricts to*

$$\mathrm{MTDer}_{\mathcal{S}}(X)^{p \leq n, \mathrm{op}} \times \mathrm{MTDer}_{\mathcal{S}}(X)^{p \geq m} \rightarrow \mathrm{MTDer}_{\mathcal{S}}(X)^{p \geq m-n} \quad (123)$$

6. *For j smooth $j^!$ and j^* are t -exact;*
7. *For j proper $j_!$ and j_* are t -exact.*

Proof. See [BBD82, Propositions 2.1.6 and 2.1.20]. \square

Here we encounter a technical difficulty. For étale sheaves Artin's vanishing theorem [AGV71, XIV, Theorem 3.1] implies that affine maps are exact with respect to the perverse t -structure. In particular for an affinely stratified variety the *standard objects* $\nabla_s := j_* \mathbb{1}_{X_s}[\dim X_s]$ and equivalently *costandard objects* $\Delta_s := j_! \mathbb{1}_{X_s}[\dim X_s]$ are perverse. There is no motivic proof of this fact yet, and hence we have to add this to our assumptions. It will be however fulfilled for the flag variety.

Definition 4.8. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and suppose all standard objects are perverse. That is, suppose that for every stratum $j : X_s \rightarrow X$, the *standard object* $\Delta_s = j_* \mathbb{1}_{X_s}[\dim X_s]$ is perverse. We say that $E \in \mathrm{Per}_{\mathcal{S}}(X)$ has

1. a *standard flag* if E has a filtration whose subquotients are standard objects $\nabla_s(a)$ for $s \in \mathcal{S}, a \in \mathbb{Z}$ or
2. a *costandard flag* if E has a filtration whose subquotients are costandard objects $\Delta_s(a)$ for $s \in \mathcal{S}, a \in \mathbb{Z}$.

Proposition 4.9. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. Then $\mathrm{Per}_{(\mathcal{S})}(X)$ has enough projective and injective objects. Furthermore the projective objects have a costandard flag and the injective objects have a standard flag.*

Proof. See [BGS96, Theorem 3.2.1] and [SW16, Proposition 11.7]. \square

Lemma 4.10. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. Let $E, F \in \mathrm{Per}_{\mathcal{S}}(X)$ such that E has costandard flag and F has a standard flag. Then for all $n \neq 0$ we have $\mathrm{Hom}_{\mathrm{H}(X)}(E, F[n]) = 0$.*

Proof. See for example [SW16, Lemma 11.8, Theorem 11.10]. \square

These statements allow us to apply Keller's dg-tilting.

Theorem 4.11. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. Then tilting induces the following equivalences of categories*

$$\mathrm{Der}^b(\mathrm{Per}_{\mathcal{S}}(X)) \xrightarrow{\sim} \mathrm{Hot}^b(\mathrm{Proj}(\mathrm{Per}_{\mathcal{S}}(X))) \xrightarrow{\sim} \mathrm{MTDer}_{\mathcal{S}}(X). \quad (124)$$

Proof. Lemma 4.10 implies that the projective modules in $\mathrm{Per}_{\mathcal{S}}(X)$ form a tilting collection and the statement follows as in 3.16. \square

In the preceding theorem we can replace projective objects by so called tilting objects, which are not to be confused with the notion of a tilting collection in the sense of Keller, but rather get their name from the fact that they form a tilting collection. For a nice reference see [BBM04].

Definition 4.12. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. An object $E \in \mathrm{Per}_{\mathcal{S}}(X)$ is called *tilting* if it has both a standard and costandard flag. We denote the additive subcategory of *tilting perverse motives* by $\mathrm{Tilt}(\mathrm{Per}_{\mathcal{S}}(X))$.

Proposition 4.13. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. Then for every stratum $j : X_s \rightarrow X$ there exists a unique tilting perverse sheaf T_s supported on \overline{X}_s with $j^*T_s = \mathbb{1}_{X_s}[\dim X_s]$.*

Proof. This follows from standard theory of highest weight categories and Proposition 4.9 and Lemma 4.10. See for example [Rin89, Section 4/5]. \square

Again we can apply tilting to obtain:

Theorem 4.14. *Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and assume that all standard objects are perverse. Then tilting induces an equivalence of categories*

$$\mathrm{Hot}^b(\mathrm{Tilt}(\mathrm{Per}_{\mathcal{S}}(X))) \xrightarrow{\sim} \mathrm{MTDer}_{\mathcal{S}}(X). \quad (125)$$

Proof. This works exactly as in the proof of Theorem 3.16 while using Lemma 4.10. Also see [BBM04, Proposition 1.5.] for a proof in the non-mixed setting. \square

4.3 Example: The flag variety

Denote by $G \supset B \supset T$ a split reductive algebraic group over $\overline{\mathbb{F}}_p$ with a Borel subgroup B and maximal torus T . Denote by $X(T) \supset \Phi \supset \Phi_+$ the character lattice, root system, and positive roots associated to $B \supset T$. Let $X = G/B$ denote the *flag variety* of G . Write $W = N_G(T)/T$ for the Weyl group with simple reflections S and $X_w = BwB/B$ for the Bruhat cell for $w \in W$. Then the Bruhat decomposition

$$X = \bigcup_{w \in W} X_w, \quad (126)$$

gives rise to an affine Whitney Tate stratification, which is denoted $(X, (B))$.

In the following, we will study the category $\text{MTDer}_{(B)}(X)$ and certain interesting subcategories of it.

4.3.1 Parity motives and Soergel modules

We start by describing some motivic cohomology rings. To the root system Φ of G one associates a positive integer t_Φ , called the *torsion index*, see [Dem73] and [Gro58]. The torsion index t_Φ is a product of primes associated to the simple constituents of Φ , which can be found in the following table.

| | | | |
|------------|---|-----------------|---------|
| A_l, C_l | $B_l \ (l \geq 3), D_l \ (l \geq 4), G_2$ | E_6, E_7, F_4 | E_8 |
| 1 | 2 | 2, 3 | 2, 3, 5 |

For an arbitrary \mathbb{Z} -module M , denote by $S(M)$ its symmetric algebra, where we by convention put M in degree 2. The *coinvariant algebra* is defined by

$$C \stackrel{\text{def}}{=} \left(S(X(T)) / S(X(T))_+^W \right) \otimes \mathbb{k}, \quad (127)$$

where $X(T) = \text{Hom}_{\text{Sch}(\mathbb{F}_p)}(T, \mathbb{G}_m) \cong \mathbb{Z}^{\text{rank}(T)}$ denotes the character lattice and $S(X(T))_+^W$ denotes W -invariant elements of degree greater than zero.

Theorem 4.15. *Assume that t_Φ is invertible in \mathbb{k} . Then there is an isomorphism of graded \mathbb{k} -algebras*

$$C \xrightarrow{\sim} \mathbb{H}(X, \mathbb{k}). \quad (128)$$

Let $s \in S$ be a simple reflection and $P_s = BsB \cup B$ the associated minimal parabolic. Then furthermore

$$C^s \cong \mathbb{H}(G/P_s, \mathbb{k}). \quad (129)$$

Proof. Since X is an affinely stratified smooth variety we can use Theorem 3.19 to identify $\mathbb{H}(X)$ with the classical Chow ring of X with coefficients in \mathbb{k} . Under our hypotheses, [Dem73, Section 8] and [Dem74] shows that the first Chern class

$$X(T) \rightarrow \text{Pic } X \xrightarrow{c_1} CH^1(X) \quad (130)$$

induces the claimed isomorphism.

For the second statement, again $\mathbb{H}(G/P_s)$ can be identified with the classical Chow ring. Denote by $L \subset P$ the Levi subgroup of $P = P_s$. Then by [Kri13, Prop.3.4 and Cor.5.9] we have the following chain of equalities

$$CH^*(G/P) = CH_P^*(G) = CH_L^*(G) = (CH_T^*(G))^{W_L} = (CH^*(G/B))^{W_L}, \quad (131)$$

where the subscript denotes equivariant Chow groups and $W_L = \{1, s\}$ is the Weyl group of L . \square

Our next goal is to understand $\mathrm{MTDer}_{(B)}(X)_{w=0}$ using the ideas and results of [Soe00]. For $w \in W$, let $w = s_1 \dots s_l$ with $s_i \in S$ be a reduced expression which we will write as $\underline{w} = (s_1, \dots, s_l)$. Recall the *Bott-Samelson resolution* of the *Schubert variety* \overline{X}_w given by

$$\pi_{\underline{w}} : \mathrm{BS}(\underline{w}) \stackrel{\mathrm{def}}{=} P_{s_1} \times^B \dots \times^B P_{s_l} / B \rightarrow \overline{X}_w \subset X, \quad (132)$$

where the morphism is given by multiplication. The variety $\mathrm{BS}(\underline{w})$ is smooth and $\pi_{\underline{w}}$ is proper and induces an isomorphism on X_w . Hence, we can apply Theorem 4.4 and the Erweiterungssatz (Theorem 3.20) to identify weight zero motives, weight zero parity motives and Soergel modules:

$$\begin{aligned} \mathrm{MTDer}_{(B)}(X, \mathbb{k})_{w=0} &= \mathrm{Par}_{(B)}(X, \mathbb{k})_{w=0} \\ &= \langle \pi_{\underline{w},!} \mathbb{1}_{\mathrm{BS}(\underline{w})}(n)[2n] \mid w \in W, n \in \mathbb{Z} \rangle_{\oplus, \epsilon, \cong} \\ &\cong \langle \mathbb{H}(\pi_{\underline{w},!} \mathbb{1}_{\mathrm{BS}(\underline{w})}(n)[2n]) \mid w \in W, n \in \mathbb{Z} \rangle_{\oplus, \epsilon, \cong} \\ &\subset \mathrm{mod}^{\mathbb{Z}}\text{-}\mathbb{H}(X, \mathbb{k}). \end{aligned}$$

Here \oplus , ϵ and \cong means closure under finite direct sums, direct summands and isomorphisms in the category of motives and $\mathbb{H}(X)$ -modules, respectively.

As a last step we want to recall Soergel's explicit description of the *Bott-Samelson modules* $\mathbb{H}(\pi_{\underline{w},!} \mathbb{1}_{\mathrm{BS}(\underline{w})})$. For $s \in S$ denote by $\pi_s : X = G/B \rightarrow G/P_s$ the projection. Then by [Soe00, Lemma 3.2.1] we have

$$\pi_{\underline{w},!} \mathbb{1}_{\mathrm{BS}(\underline{w})} = \pi_{s_l}^* \pi_{s_l,*} \cdots \pi_{s_1}^* \pi_{s_1,*} \mathbb{1}_{B/B}. \quad (133)$$

Hence we need to understand the interaction of the functors \mathbb{H} and $\pi_s^* \pi_{s,*}$.

Lemma 4.16. *Let $s \in S$ be a simple reflection. Then*

$$\pi_{s,*} \mathbb{1}_{G/B} = \mathbb{1}_{G/P_s} \oplus \mathbb{1}_{G/P_s}(-1)[-2].$$

Assume that t_R is invertible in \mathbb{k} . Then there is a natural equivalence of functors

$$\mathbb{H}(\pi_{s,*} \pi_s^*(-)) \cong C \otimes_{C^s} \mathbb{H}(-) : \mathrm{MTDer}_{(B)}(X)_{w=0} \rightarrow C\text{-mod}^{\mathbb{Z}}.$$

Proof. There are two different proofs for the first statement which do not apply in our situation. The first one given in [Soe90] uses the decomposition theorem for perverse sheaves and the second one in [Soe00] relies on a concrete description of the category of sheaves and the identification $G(\mathbb{C})/B(\mathbb{C}) = K/T$ for a compact real form K of the complex group $G(\mathbb{C})$, while it also requires that 2 is invertible in \mathbb{k} . But one can also apply the projective bundle formula (see Proposition 2.54). It is well known that $\pi_s : G/B \rightarrow G/P_s$ is a \mathbb{P}^1 -bundle. By [Har77, Exercise 7.10(c)] this bundle is the projectivization $\mathbb{P}(\mathcal{E})$ of a vector bundle \mathcal{E} on G/P_s , since G/B

is regular and Noetherian. In our case there is an completely explicit description of the bundle \mathcal{E} for which we did not find an reference. Hence the projective bundle formula applies and the first statement follows.

Following [Soe90, Theorem 14] or [Soe00, Proposition 4.1.1] this implies that there is a natural isomorphism of functors

$$\mathbb{H}(\pi_{s*} \pi_s^*(-)) \cong \mathbb{H}(X) \otimes_{\mathbb{H}(Y)} \mathbb{H}(-) : \mathrm{MTDer}_{(B)}(X) \rightarrow \mathbb{H}(X)\text{-mod}^{\mathbb{Z}}$$

and the statement follows using Theorem 4.15. \square

Using $\mathbb{H}(\mathbb{1}_{B/B}) = \mathbb{k}$ and applying the preceding Lemma we get

$$\mathbb{H}(\pi_{w*} \mathbb{1}_{\mathrm{BS}(w)}) \cong C \otimes_{C^{s_l}} \cdots C \otimes_{C^{s_1}} \mathbb{k}. \quad (134)$$

Furthermore, the pointwise purity of the motives $\pi_{w*} \mathbb{1}_{\mathrm{BS}(w)}$ allows us to use the tilting result (Theorem 3.16). So in conclusion we obtain the following theorem.

Theorem 4.17. *There is an equivalence of categories*

$$\mathrm{MTDer}_{(B)}(X) = \mathrm{Hot}^b(\mathrm{Par}_{(B)}(X)_{w=0}). \quad (135)$$

Assume that t_Φ is invertible in \mathbb{k} . Then $\mathrm{Par}_{(B)}(X)_{w=0}$ can be identified with the category of evenly graded Soergel modules

$$C\text{-Smod}_{ev}^{\mathbb{Z}} = \langle C \otimes_{C^{s_1}} \cdots C \otimes_{C^{s_n}} \mathbb{k} \mid s_i \in S \rangle_{\oplus, \otimes, \cong, \langle 2- \rangle} \quad (136)$$

where \oplus, \otimes, \cong and $\langle 2- \rangle$ means closure under finite direct sums, direct summands, isomorphisms and even shifts of grading in the category of graded C -modules.

Under this isomorphism the unique indecomposable parity motive E_w with $j_w^* E_w = \mathbb{1}_{X_w}$ gets identified with the unique indecomposable Soergel module D_w which appears as a direct summand of $C \otimes_{C^{s_1}} \cdots C \otimes_{C^{s_n}} \mathbb{k}$ but not in the corresponding modules for smaller expressions.

Remark 4.18. The equivalence $\mathrm{Par}_{(B)}(X) \xrightarrow{\sim} C\text{-Smod}_{ev}^{\mathbb{Z}}$ also proves that in case of the flag variety, the category of stratified mixed Tate motives is equivalent to the *mixed derived category* as considered in [AR16b, Definition 2.1]. Their mixed derived category is by construction the homotopy category of Soergel modules.

4.3.2 Perverse motives

We start by showing that the technical requirements for a nice theory of perverse motives and tilting perverse motives are in fact met by the flag variety.

Lemma 4.19. *For all $w \in W$ and $j : X_w \rightarrow X$ the standard objects $\nabla_w = j_* \mathbb{1}_{X_s}[\dim X_s]$ and costandard objects $\Delta_w = j_! \mathbb{1}_{X_s}[\dim X_s]$ are perverse.*

Proof. By duality it suffices to prove the first statement. By Proposition 4.7 we know that $\nabla_w \in \mathrm{MTDer}_{(B)}(X)^{p \leq 0}$. To show that $\nabla_w \in \mathrm{MTDer}_{(B)}(X)^{p \geq 0}$ we proceed by induction on the length of w . If $w = e$, then j is a closed embedding. Hence $j_! = j_*$ and the statement follows from Proposition 4.7. Otherwise, let $s \in W$ be such that $ws < w$ and let $\pi : G/B \rightarrow G/P_s$ be the projection. Then we obtain the distinguished triangle

$$\nabla_{ws} \longrightarrow \pi^! \pi_! \nabla_{ws} \longrightarrow \nabla_w(1)[1] \xrightarrow{+1} \quad (137)$$

where ∇_{ws} is perverse by induction. Now let $x \in W$ and assume that $xs > s$. Then we obtain the cartesian square

$$\begin{array}{ccc} X_{xs} \cup X_x & \xrightarrow{k} & X \\ p \downarrow & & \downarrow \pi \\ BxP_s/P_s & \xrightarrow{i} & G/P_s. \end{array}$$

Applying base change and k^* our distinguished triangle becomes

$$k^* \nabla_{ws} \longrightarrow p^! p_! k^* \nabla_{ws} \longrightarrow k^* \nabla_w(1)[1] \xrightarrow{+1} \quad (138)$$

Now p is a trivial \mathbb{P}^1 -bundle. Hence, we have reduced our statement to the case $X = \mathbb{P}^1$ where it follows easily. \square

Hence Section 4.2 implies the following equivalent descriptions of the category of stratified mixed Tate motives on X .

Theorem 4.20. *There are equivalences of categories*

$$\begin{aligned} \mathrm{Hot}^b(\mathrm{Tilt}(\mathrm{Per}_{(B)}(X))) &\xleftarrow{\sim} \mathrm{MTDer}_{(B)}(X) \xrightarrow{\sim} \mathrm{Hot}^b(\mathrm{Proj}(\mathrm{Per}_{(B)}(X))) \\ &\xrightarrow{\sim} \mathrm{Der}^b(\mathrm{Per}_{(B)}(X)). \end{aligned}$$

4.3.3 Ringel duality and Radon transform

For a quasi-hereditary algebra A , *Ringel duality* is an equivalence between the category of modules over A and its *Ringel dual* algebra B . It exchanges tilting and projective/injective objects. In the case of the flag variety it is in fact a self equivalence and has a geometric interpretation called the *Radon transform*. The Radon transform only requires a six functor formalism and hence immediately applies in our setting. A good reference for this is [AR16b, Appendix B] or [BBM04]. Denote by $U \subset X \times X$ the open G -orbit and consider the following diagram.

$$\begin{array}{ccccc} & & X \times X & & \\ & \swarrow pr_1 & \uparrow & \searrow pr_2 & \\ X & \xleftarrow{\overleftarrow{u}} & U & \xrightarrow{\overrightarrow{u}} & X \end{array}$$

Then the Radon transform is defined as

$$R \stackrel{\text{def}}{=} \overrightarrow{u} \circ \overleftarrow{u}^*[\dim X] : \text{MTDer}_{(B)}(X) \rightarrow \text{MTDer}_{(B)}(X).$$

It has the following properties.

Theorem 4.21. *The functor R is a self equivalence of $\text{MTDer}_{(B)}(X)$. Furthermore*

$$\begin{aligned} R(\nabla_w) &= \Delta_{ww_0}, \\ R(T_w) &= P_{ww_0}, \\ R(I_w) &= T_{ww_0}, \end{aligned}$$

where T_w, P_w and I_w denote the indecomposable tilting, projective, respectively injective perverse motive corresponding to X_w . It hence induces an equivalence of categories

$$R : \text{Tilt}(\text{Per}_{(B)}(X)) \rightarrow \text{Proj}(\text{Per}_{(B)}(X)). \quad (139)$$

Proof. See [BBM04]. \square

5 Representation Theory

In this section we apply our results to the representation theory of semisimple algebraic groups in characteristic p .

5.1 Modular Category \mathcal{O}

Let $G \supset B \supset T$ be a split semisimple simply connected algebraic group with a Borel subgroup and maximal torus over a field \mathbb{k} of characteristic p . Assume that p is bigger than the Coxeter number of G . Denote by $N_G(T)/T = W \supset \mathcal{S}$ the corresponding Weyl group and simple reflections and by $X(T) \supset \Phi \supset \Phi_+ \supset \Delta$ the associated root lattice, root system, positive and simple roots.

For $\lambda \in X(T)$ dominant write $\text{Ind}_B^G \mathbb{k}_\lambda = H^0(G/B, \mathcal{O}(\lambda)) = H^0(\lambda)$ for the induced representation of the one-dimensional T -module \mathbb{k}_λ . Over the complex numbers, those are exactly all simple rational representations (Borel–Weil–Bott Theorem) and they have a nice character formula (Weyl character formula). In positive characteristic though, the modules $H^0(\lambda)$ can become reducible and the main goal is to determine their composition factors. For astronomically big (see [Fie12]) prime numbers p , this is solved by the proof of the Lusztig conjecture in [AJS94], which turns out to be false for smaller primes, as shown by [Wil13] using the modular category \mathcal{O} .

The modular category $\mathcal{O}(G, B)$, also called *subquotient around the Steinberg point*, is a subquotient of the category G -mod of finite dimensional representations

of G over \mathbb{k} . It was defined by Soergel [Soe90] in the following way. In the notation of [Jan03], let $L(\lambda)$ denote the unique simple submodule of $H^0(\lambda)$ and let st and ρ denote the Steinberg weight and half sum of all positive roots. Soergel then defines two full subcategories of $G\text{-mod}$ by

$$\mathcal{A} = \{M \in G\text{-mod} \mid [M : L(\lambda)] \neq 0 \Rightarrow \lambda \uparrow \text{st} + \rho\} \text{ and} \quad (140)$$

$$\mathcal{N} = \{M \in G\text{-mod} \mid [M : L(\lambda)] = 0 \Rightarrow \lambda \in \text{st} + W\rho\}. \quad (141)$$

Here $[M : L(\lambda)]$ is the number of times $L(\lambda)$ appears as factor in a composition series of M , and $\lambda \uparrow \text{st} + \rho$ means that λ is linked to $\text{st} + \rho$, with respect to the p -dilated action of the affine Weyl-group (see [Jan03, Chapter 6, The Linkage Principle]).

Definition 5.1. The modular category $\mathcal{O}(G, B)$ is then the quotient

$$\mathcal{O}(G, B) \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{N}. \quad (142)$$

The modular category $\mathcal{O}(G, B)$ resembles the BGG category $\mathcal{O}_0(\mathfrak{g})$ associated to complex reductive Lie algebras \mathfrak{g} (see [BGG71]) in many ways. It has standard objects $M_x = H^0(\text{st} + x\rho)^*$ with unique simple quotient L_x and projective covers P_x , all parametrized by elements of the Weyl group $x \in W$. In a way, it is a *window into* or *excerpt of* the category of all finite dimensional representations, which can be used to test or prove conjectures with methods used in the study of category $\mathcal{O}_0(\mathfrak{g})$. Indeed, it was introduced by Soergel in [Soe00] in the hope to partly prove the Lusztig conjecture and the mentioned counterexamples by Williamson [Wil13] are constructed using the modular category \mathcal{O} .

The modular category also has an analogue to the *Struktur-* and *Endomorphismensatz* from [Soe90].

Theorem 5.2 ([AJS94, 19.8], [Soe00, Theorem 2.6.1]). *The functor*

$$\mathbb{V} \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}}(P_{w_0}, -) : \mathcal{O} \rightarrow \text{mod-End}_{\mathcal{O}}(P_{w_0}) \quad (143)$$

is fully faithful on projective modules. Here w_0 denotes the longest word in W . Furthermore

$$\text{End}_{\mathcal{O}}(P_{w_0}) = C \stackrel{\text{def}}{=} S(\mathfrak{h})/S(\mathfrak{h})_+^W. \quad (144)$$

where $\mathfrak{h} = \text{Lie}(T)$, and $S(\mathfrak{h})$ denotes the symmetric algebra.

Moreover, by analysing the interaction of the functor \mathbb{V} with translation functors, [Soe00] identifies the essential image of the projective modules in \mathcal{O} under \mathbb{V} with the category $C\text{-Smod}$ of *Soergel modules*.

Theorem 5.3 ([Soe00] Theorem 2.8.2.). *The essential image of \mathbb{V} is the category of Soergel modules*

$$C\text{-Smod} \stackrel{\text{def}}{=} \langle C \otimes_{C^{s_1}} \cdots C \otimes_{C^{s_n}} \mathbb{k} \mid s_i \in S \rangle_{\oplus, \oplus, \cong}$$

where \oplus , \in and \cong means closure under finite direct sums, direct summands and isomorphisms in the category of C -modules.

Putting these results together we get a *combinatorial* description of the derived modular category \mathcal{O} in terms of the homotopy category of Soergel modules

$$\mathrm{Der}^b(\mathcal{O}) \xrightarrow{\sim} \mathrm{Hot}^b(\mathrm{Proj} \mathcal{O}) \xrightarrow[\sim]{\mathbb{V}} \mathrm{Hot}^b(C\text{-}\mathrm{Smod}).$$

We now combine this with the results from Section 4.3. Let $G^\vee \supset B^\vee \supset T^\vee$ be a semisimple algebraic group over $\overline{\mathbb{F}}_p$ with a Borel subgroup and maximal torus and root system $X(T^\vee) = Y(T) \supset \Phi^\vee$ dual to that of G . Denote by $X^\vee = G^\vee/B^\vee$ the flag variety. Under the assumption that t_{ϕ^\vee} is invertible in \mathbb{k} , Theorem 4.15 gave us a description of the motivic cohomology ring of X

$$\mathbb{H}(X) = \left(S(X(T^\vee)) / S(X(T^\vee))_+^W \right) \otimes \mathbb{k} = S(\mathfrak{h}) / S(\mathfrak{h})_+^W = C$$

and Theorem 4.17 provided us with a combinatorial description of the category of stratified mixed Tate motives on X^\vee

$$\mathrm{MTDer}_{(B^\vee)}(X^\vee) \xrightarrow{\sim} \mathrm{Hot}^b(\mathrm{Par}_{(B^\vee)}(X^\vee)) \xrightarrow[\sim]{\mathbb{H}} \mathrm{Hot}^b(C\text{-}\mathrm{Smod}_{ev}^{\mathbb{Z}}).$$

Putting everything together, we obtain our final theorem.

Theorem 5.4. *The functor induced by forgetting the grading of Soergel modules*

$$\mathrm{MTDer}_{(B^\vee)}(X^\vee, \mathbb{k}) \xrightarrow{v} \mathrm{Der}^b(\mathcal{O}(G, B))$$

has the following properties:

1. *There is natural isomorphism $v \cong v \circ (1)[2]$.*
2. *For all $E, F \in \mathrm{MTDer}_{(B^\vee)}(X^\vee, \mathbb{k})$*

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{H(X^\vee)}(E, F(n)[2n]) = \mathrm{Hom}_{\mathrm{Der}^b(\mathcal{O})}(v(E), v(F)).$$

3. *For every indecomposable projective module P_x in \mathcal{O} there is an indecomposable pointwise pure parity motive $E_x \in \mathrm{Par}_{(B^\vee)}(X^\vee)_{w=0}$ with $j_x^* E_x = \mathbb{1}_{X_s^\vee}$, such that*

$$v(E_x) = P_x.$$

4. *The constant motives on the Bruhat cells correspond to standard modules*

$$v(j_{x,*} \mathbb{1}_{X_x^\vee}) = M_x.$$

Remark 5.5. We could also formulate the last theorem as an equivalence

$$\mathrm{MTDer}_{(B^\vee)}(G^\vee/B^\vee) \xrightarrow{\sim} \mathrm{Der}^b(\mathcal{O}^{\mathbb{Z}, ev})$$

as stated in the introduction, because the right hand side is by definition equivalent to $\text{Hot}^b(C\text{-Smod}^{\mathbb{Z}})$. Alternatively, one could also artificially add a root of the Tate twist on the geometric side to get an equivalence with the whole derived graded category $\text{Der}^b(\mathcal{O}^s\mathbb{Z})$.

6 Rappels: Some category theory

In this section we recall some notions from category theory which may not be so well known for the convenience of the reader.

6.1 2-functors

Definition 6.1 (cf. [Del01, §2]). A 2-functor $H : \text{QProj}/k)^{\text{op}} \rightarrow \text{Cat}$ is an assignment sending every variety $S \in \text{QProj}/k$ to a category $H(S)$, every morphism $f : T \rightarrow S$ in QProj/k to a functor $f^* : H(S) \rightarrow H(T)$, and every pair of composable morphisms $\xrightarrow{g} \xrightarrow{f}$ to a natural isomorphism $\alpha_{g,f} : g^* f^* \xrightarrow{\sim} (fg)^*$, such that for any triple of composable morphisms $\xrightarrow{h} \xrightarrow{g} \xrightarrow{f}$, the cocycle condition $\alpha_{h,gf}(h^* \alpha_{g,f}) = \alpha_{hg,f}(\alpha_{h,g} f^*)$ is satisfied.

Definition 6.2. A 2-functor H takes values in symmetric monoidal categories if each $H(S)$ is equipped with the structure of symmetric monoidal category, i.e., it has a symmetric product $\otimes_S : H(S) \times H(S) \rightarrow H(S)$, and an identity object $\mathbb{1}_S$ for the product, and the functors f^* are compatible with the monoidal structure, i.e., for every $f : T \rightarrow S$, the functor f^* is strong symmetric monoidal in the sense that there are natural isomorphisms $(f^* -) \otimes_T (f^* -) \xrightarrow{\sim} f^* (- \otimes_S -)$, and $f^* \mathbb{1}_S \xrightarrow{\sim} \mathbb{1}_T$.

Definition 6.3. A 2-functor H takes values in triangulated categories if each $H(S)$ is equipped with the structure of a triangulated category, and the functors f^* are exact.

Definition 6.4. A 2-functor H takes values in symmetric monoidal triangulated categories if it takes values in both symmetric monoidal categories, and triangulated categories, and for each $S \in \text{QProj}/k$, the functor \otimes_S is exact, i.e., compatible with the triangulated structure, in both variables.

6.2 Triangulated categories

Definition 6.5. One says that a triangulated category \mathcal{T} admits all small sums if for any set of objects $\{\mathcal{E}_i\}_{i \in I}$ the coproduct, i.e., the object corepresenting the functor $\prod_{i \in I} \text{Hom}_{\mathcal{T}}(\mathcal{E}_i, -)$, exists. It is denoted $\oplus_{i \in I} \mathcal{E}_i$, so by definition we have $\prod_{i \in I} \text{Hom}_{\mathcal{T}}(\mathcal{E}_i, -) \cong \text{Hom}_{\mathcal{T}}(\oplus_{i \in I} \mathcal{E}_i, -)$.

Definition 6.6. An object $\mathcal{F} \in \mathcal{T}$ is said to be *compact* if $\text{Hom}_{\mathcal{T}}(\mathcal{F}, -)$ commutes with sums.

Theorem 6.7. Let \mathcal{T} be a triangulated category admitting all small sums and $\{\mathcal{G}_i\}_{i \in I}$ a set of compact objects. The following are equivalent, [Nee01, Thm.8.3.3].

1. For every $\mathcal{E} \in \mathcal{T}$ we have \mathcal{E} is zero if and only if $\text{Hom}_{\mathcal{T}}(\mathcal{G}_i[n], \mathcal{E}) = 0$ for all $i \in I, n \in \mathbb{Z}$. Equivalently, a morphism f is an isomorphism if and only if $\text{Hom}_{\mathcal{T}}(\mathcal{G}_i[n], f)$ is an isomorphism for all $i \in I, n \in \mathbb{Z}$.
2. The smallest triangulated subcategory of \mathcal{T} admitting all small sums and contain the objects \mathcal{G}_i is \mathcal{T} itself.

Definition 6.8 ([Nee96, Def.1.7,1.8]). If the equivalent conditions of the above theorem are satisfied, we say that \mathcal{T} is *compactly generated* and that $\{\mathcal{G}_i[n]\}_{i \in I, n \in \mathbb{Z}}$ is a *generating set*.

Theorem 6.9 ([Nee96, Thm.4.1]). If $\phi : \mathcal{T} \rightarrow \mathcal{T}'$ is an exact functor between triangulated categories with \mathcal{T} admitting small sums and compactly generated, then ϕ preserves sums (resp. products) if and only if ϕ admits a right (resp. left) adjoint.

Remark 6.10. The left adjoint version doesn't seem to be stated in [Nee96], but it is used in [Nee96, Exam.4.2,4.3], and one sees easily that the proof of [Nee96, Thm.4.1] also works to produce a left adjoint.

Definition 6.11. The *Verdier quotient* of a triangulated category \mathcal{T} by a collection of objects \mathcal{S} is a triangulated category equipped with a triangulated functor $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$ with the property that all objects in \mathcal{S} are sent to the zero object, and is universal in the sense that any other such functor factors through \mathcal{T}/\mathcal{S} uniquely up to unique natural isomorphism, [Nee01, Theo.2.1.8, Rema.2.1.9].

Rappels 6.12. Let \mathcal{T} be a triangulated category.

1. Let \mathcal{S} be a triangulated subcategory of \mathcal{T} . Then there is a description of \mathcal{T}/\mathcal{S} whose objects are the same as those of \mathcal{T} , and morphisms are equivalence classes of “hats” of morphisms $X \xleftarrow{s} \hat{X} \xrightarrow{f} \hat{Y} \xrightarrow{s} Y$ with f, s morphisms of \mathcal{T} and $\text{Cone}(s) \in \mathcal{S}$, cf. [Ver96, 2.1.7, 2.2.1], [Nee01, 2.1.11].
2. If \mathcal{T} admits all small sums and is compactly generated, and \mathcal{S} is a triangulated subcategory which also admits small sums and is generated by objects which are compact in \mathcal{T} , then the canonical quotient functor preserves small sums, and admits a right adjoint [Nee01, Exam.8.4.5].

$$\mathcal{T} \rightleftarrows \mathcal{T}/\mathcal{S}. \quad (145)$$

3. Under the above hypotheses, the right adjoint identifies \mathcal{T}/\mathcal{S} with the full subcategory of \mathcal{T} of those objects \mathcal{E} such that $\text{Hom}_{\mathcal{T}}(f, \mathcal{E})$ is an isomorphism for every f with $\text{Cone}(f) \in \mathcal{S}$, or equivalently, those objects \mathcal{E} such that $\text{Hom}_{\mathcal{T}}(\mathcal{F}, \mathcal{E}) = 0$ for every $\mathcal{F} \in \mathcal{S}$, [Nee01, Theo.9.1.16]. In symbols,

$$\mathcal{T}/\mathcal{S} \cong \{\mathcal{E} : \text{Hom}_{\mathcal{T}}(\mathcal{F}, \mathcal{E}) = 0 \ \forall \ \mathcal{F} \in \mathcal{S}\}^{\text{full}} \subset \mathcal{T}. \quad (146)$$

Remark 6.13. Sometimes it will be better to use the description (1) above, and sometimes the description (3). The fact that these two descriptions are equivalent comes from the existence of the right adjoint, which is a consequence of Brown representability (à la Neeman). I.e., it is an important theorem, and not at all formal, although the proof is “very short and simple” [Nee96, Intro.].

6.3 Descent model structures

We recall here the definitions of cofibrations and fibrations in $C(\mathbb{R}\text{-mod})$, even though in the text, we only use the definition of fibrant object, and the fact that the $t^n\mathbb{R}(X)[i-1]$ are cofibrant.

Definition 6.14 ([CD12, 5.1.11]). The class of *cofibrations* of $C(\mathbb{R}\text{-mod})$ is the smallest class of morphisms closed under retracts, pushouts, and transfinite composition, and containing the morphisms

$$t^n\mathbb{R}(X)[i] \rightarrow \text{Cone}\left(t^n\mathbb{R}(X)[i] \rightarrow t^n\mathbb{R}(X)[i]\right) \quad (147)$$

for all $X \in \text{Sm}/S, i \in \mathbb{Z}, n \geq 0$. Note, the morphism $0 \rightarrow t^n\mathbb{R}(X)[i-1]$ is the pushout of this latter along $t^n\mathbb{R}(X)[i] \rightarrow 0$, so the objects $t^n\mathbb{R}(X)[i-1]$ are all cofibrant.

It follows from this that the functors $\mathbb{R}_S(X) \otimes -$ are left Quillen functors, but it seems that we never need this fact.

Definition 6.15 ([CD12, Def.5.1.9, 5.1.11]). Let \mathbb{R} be a cartesian commutative monoid of $\text{Sh}_{\text{Nis}}(\text{Sm}/-)^{\mathfrak{S}}$, such as \mathbb{K} or \mathbb{T} . An object $\mathcal{E} \in C(\mathbb{R}\text{-mod})$ is fibrant if and only if for every $i, n \geq 0$ and $X \in \text{Sm}/S$, the (cochain complex) cohomology $H^i(X, \mathcal{E}_n)$ and the Nisnevich hypercohomology $\mathbb{H}_{\text{Nis}}^i(X, \mathcal{E}_n)$ agree.

Definition 6.16. A morphism $f : \mathcal{E} \rightarrow \mathcal{F}$ in $C(\mathbb{R}\text{-mod})$ is a *fibration* if each $\mathcal{E}_n \rightarrow \mathcal{F}_n$ is surjective as a morphism of *presheaves*, and each kernel $\ker(\mathcal{E}_n \rightarrow \mathcal{F}_n)$ is fibrant.

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